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Member of the Syndicate Calcutta University.

THE ELEMENTS

OF

ALGEBRA

BY

JAMES WOOD, D. D.

FORMERLY DEAN OF ELY, AND MASTER OF ST. JOHN'S COLLEGE, CAMBRIDGE.

RE-MODELLED, SIMPLIFIED, AND ADAPTED FOR GENERAL USE IN SCHOOLS AND COLLEGES—WITH NUMEROUS EXERCISES, EXAMPLES AND CALCUTTA, BOMBAY AND MADRAS UNIVERSITY EXAMINATION-PAPERS.

PART II.

FOR THE CANDIDATES OF THE INDIAN UNIVERSITIES.

BY

P. GHOSH.

AUTHOR OF "ELEMENTS OF ARITHMETIC," "ELEMENTS OF MENSURATION", "ELEMENTS OF EUCLID", "PLANE TRIGONOMETRY", ETC., ETC,

NEW EDITION.

(Revised and Enlarged.)

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CONTENTS.

THE

ELEMENTS OF ALGEBRA.

PART II.

QUADRATIC EQUATIONS.

183. A QUADRATIC EQUATION, or an equation of two dimensions, is that into which the second power of the unknown quantity enters, *without* or *with* the first power. Thus $x^2=25$ and $x^2+2x=15$ are quadratic equations.

Pure Quadratic Equations.

184. When the terms of an equation involve the square of the unknown quantity, but the first power does not appear, the equation is called a *pure quadratic equation*, and the value of the square is obtained by the rules for solving simple equations; and by extracting the square root of both sides, the quantity itself is found.

Ex. 1. $5x^2-45=0$; to find x.

By transp. $5x^2=45$.

$$x^2=9.$$

$$\therefore \text{ (Art. 113), } x=\sqrt{9}=\pm 3.$$

The signs $+$ and $-$ are both prefixed to the root, because the square root of a quantity may be either positive or negative (Art. 93). The sign of x may also be negative; but still x will be either equal to $+3$ or -3.

Ex. 2. $ax^2=bcd$; to find x.

$$x^2=\frac{bcd}{a} \quad \therefore\ x=\pm\sqrt{\left(\frac{bcd}{a}\right)}.$$

Ex. 3. $\dfrac{3x+\sqrt{(9x^2-4)}}{3x-\sqrt{(9x^2-4)}}=4$; to find x.

Apply Art. 148, then $\dfrac{3x}{\sqrt{(9x^2-4)}}=\dfrac{4+1}{4-1}=\dfrac{5}{3}$.

Squaring $$\frac{9x^2}{9x^2-4}=\frac{25}{9}$$

$$\therefore \frac{9x^2}{4}=\frac{25}{16}; \quad \text{(Art. 147).}$$

$$\therefore x^2=\tfrac{25}{36}; \therefore x=\pm\tfrac{5}{6}.$$

Ex. 4. $\dfrac{1}{1-\sqrt{(1-x^2)}}-\dfrac{1}{1+\sqrt{(1-x^2)}}=\dfrac{1}{x^2}$; to find x.

Simplifying the left-hand side, we have

$$\frac{1+\sqrt{(1-x^2)}-1+\sqrt{(1-x^2)}}{1-(1-x^2)}=\frac{1}{x^2}; \text{ or } \frac{2\sqrt{(1-x^2)}}{x^2}=\frac{1}{x^2};$$

$\therefore 2\sqrt{(1-x^2)}=1$; squaring, we have

$$4-4x^2=1; \therefore 4x^2=3; \therefore x=\pm\tfrac{1}{2}\sqrt{3}.$$

Ex. 5. $x+\sqrt{(x^2-a^2)}=\dfrac{na^2}{\sqrt{(x^2-a^2)}}$; to find x.

Multiply each side of the equation by $\sqrt{(x^2-a^2)}$, then

$$x\sqrt{(x^2-a^2)}+x^2-a^2=na^2$$

$$\therefore x\sqrt{(x^2-a^2)}=(n+1)a^2-x^2.$$

Squaring both sides of the equation, we have

$$x^4-a^2x^2=(n+1)^2a^4-2a^2x^2(n+1)+x^4;$$

$$a^2x^2(2n+1)=(n+1)^2a^4;$$

$$\therefore x^2=\frac{(n+1)^2a^4}{(2n+1)a^2}=\frac{(n+1)^2a^2}{2n+1}$$

$$\therefore x=\pm\frac{(n+1)a}{\sqrt{(2n+1)}}.$$

Ex. 6. $\dfrac{x}{\sqrt{(x+1)}-1}+\dfrac{x-2}{\sqrt{(x-1)}+1}=\sqrt{(2x+1)}$; find x.

$$\frac{(x+1)-1}{\sqrt{(x+1)}-1}+\frac{(x-1)-1}{\sqrt{(x-1)}+1}=\sqrt{(2x+1)};$$

$$\therefore \frac{\{\sqrt{(x+1)}+1\}\{\sqrt{(x+1)}-1\}}{\sqrt{(x+1)}-1}+\frac{\{\sqrt{(x-1)}-1\}\{\sqrt{(x-1)}+1\}}{\sqrt{(x-1)}+1}$$

$$=\sqrt{(2x+1)}.$$

$$\therefore \quad \sqrt{(x+1)}+1+\sqrt{(x-1)}-1=\sqrt{(2x+1)}.$$

$$\therefore \quad \sqrt{(x+1)}+\sqrt{(x-1)}=\sqrt{(2x+1)}.$$

$$\therefore \quad x+1+x-1+2\sqrt{(x^2-1)}=2x+1$$

$$\therefore \quad 2\sqrt{(x^2-1)}=1$$

$$4x^2-4=1$$

$$\therefore \quad 4x^2=5\ ;\ x=\pm\tfrac{1}{2}\sqrt{5}.$$

Otherwise :—Rationalizing the denominators, we have

$$\frac{x\{\sqrt{(x+1)}+1\}}{x}+\frac{(x-2)\{\sqrt{(x-1)}-1\}}{x-2}=\sqrt{(2x+1)}.$$

$$\therefore \quad \sqrt{(x+1)}+1+\sqrt{(x-1)}-1=\sqrt{(2x+1)}\ ;\ \text{etc.}$$

EXAMPLES LXV.

Solve the following equations,

1. $(x+3)^2=6x+25.$ 2. $7x^2-8\tfrac{3}{4}=7\tfrac{1}{4}+3x^2.$

3. $\dfrac{10x^2+17}{18}-\dfrac{12x^2+2}{11x^2-8}=\dfrac{5x^2-4}{9}.$

4. $\dfrac{30x^2+13}{15}-\dfrac{3x^2+1}{5(x^2-1)}=2x^2.$ 5. $\dfrac{2}{x^2}+\dfrac{3}{x^2}+\dfrac{4}{x^2}+\dfrac{5}{x^2}=56.$

6. $\dfrac{2}{x^2+1}+\dfrac{5}{2x^2+2}+\dfrac{6x^2-6}{x^4-1}=2\tfrac{1}{10}.$

7. $\dfrac{14x^2+16}{21}-\dfrac{2x^2+8}{8x^2-11}=\dfrac{2x^2}{3}.$ 8. $\dfrac{x+3}{x+2}+\dfrac{x-3}{x-2}=\dfrac{5}{3}.$

9. $\sqrt{(x^2+13)}=2+\sqrt{(x^2-11)}.$

10. $x+\sqrt{\{a^2+x^2-x\sqrt{(a^2+x^2)}\}}=a.$

11. $\dfrac{1}{1-\sqrt{(1-x^2)}}-\dfrac{1}{1+\sqrt{(1-x^2)}}=\dfrac{1}{x}.$

✓ 12. $\dfrac{\sqrt{(1+x)}}{1+\sqrt{(1+x)}}+\dfrac{\sqrt{(1-x)}}{1+\sqrt{(1-x)}}=0.$

13. $\sqrt[3]{(1+x)}+\sqrt[3]{(1-x)}=\sqrt[3]{2}.$

14. $x+\sqrt{(x^2+a^2n^2-a^2)}=\dfrac{a^2n(n+1)}{\sqrt{(x^2-a^2+a^2n^2)}}$.

15. $\dfrac{x-3}{\sqrt{(x+1)}-2}+\dfrac{x-2}{\sqrt{(x-1)}+1}=\dfrac{2x}{\sqrt{(2x+1)}-1}$.

16. $\dfrac{x}{\sqrt{(x+a)}-\sqrt{a}}+\dfrac{x-2a}{\sqrt{(x-a)}+\sqrt{a}}=\sqrt{(2x+a.)}$.

17. $\dfrac{x+1}{\sqrt{(x+2)}+1}+\dfrac{x+2}{\sqrt{(x+3)}-1}=\sqrt{\{2x+5+2\sqrt{(5x+7)}\}}$.

18. $\dfrac{x-3}{\sqrt{(x+1)}-2}+\dfrac{x-2}{\sqrt{(x-1)}+1}=\dfrac{2x+1}{\sqrt{2\sqrt{(x+1)}-1}}$.

19. $\dfrac{x^2-\sqrt[n]{(x^{2n}-a^{2n})}}{x^2-\sqrt[n]{(x^{2n}-a^{2n})}}=\dfrac{b^2+1}{b^2-1}$.

20. $\dfrac{1}{x^2}\sqrt[n]{(a^2+x^2)}+\dfrac{1}{a^2}\sqrt[n]{(a^2+x^2)}=b^2\sqrt[n]{(x^2)}$.

21. $\dfrac{x+\sqrt{(x^2-a^2)}}{x-\sqrt{(x^2-a^2)}}+\dfrac{x-\sqrt{(x^2-a^2)}}{x+\sqrt{(x^2-a^2)}}=\dfrac{82}{9}$.

22. $\dfrac{a^2+\sqrt{(a^4-x^4)}}{a^2-\sqrt{(a^4-x^4)}}=b^{\frac{2m+1}{m}}\sqrt[m]{\left\{\dfrac{\sqrt{(a^2+x^2)}-\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}}\right\}}$.

Answers {

Adfected Quadratic Equations.

185. If both the first and second powers of the unknown quantity be found in an equation, arrange the terms according to the dimensions of the unknown quantity, beginning with the highest, and transpose the known quantities to the other side; then, if the square of the unknown quantity be affected with a coe-fficient, divide all the terms by this coe-fficient, and if its sign be negative, change the signs of all the terms (Art. 109) that the equation may be reduced to one of the forms, $x^2\pm px=\pm q$. Then add to both sides the square of half the coe-fficient of the first power of the unknown quantity, by which means, the first side of the equation is made a complete square (Art. 98) and the other consists of known quantities; and by extracting the square root of both sides, a simple equation is obtained, from which the value of the unknown quantity may be found.

Ex. 1. Let $x^2+px=q$; now, we know that $x^2+px+\frac{p^2}{4}$ is the square of $x+\frac{p}{2}$ (Art. 98); add therefore $\frac{p^2}{4}$ to both sides, and we have, $x^2+px+\frac{p^2}{4}=q+\frac{p^2}{4}$.

Then by extracting the square root of both sides,

$$x+\frac{p}{2}=\pm\sqrt{\left(q+\frac{p^2}{4}\right)}.$$

By transp. $$x=-\frac{p}{2}\pm\sqrt{\left(q+\frac{p^2}{4}\right)}.$$

In the same manner, if $x^2-px=q$,

$$x=\frac{p}{2}\pm\sqrt{\left(q+\frac{p^2}{4}\right)}.$$

Ex. 2. Let $x^2-12x+35=0$, to find x.

By transp. $x^2-12x=-35$; and adding the square of $\frac{12}{2}$ or 6, to both sides of the equation,

$$x^2-12x+36=36-35=1.$$

Then extracting the square root of both sides.

$$x-6=\pm 1$$

$$\therefore\ x=6\pm 1=7 \text{ or } 5.$$

Either of which, substituted for x in the original equation, answers the condition, that is, makes the whole expression $x^2-12x+35$ equal to nothing.

Ex. 3. Let $\frac{6}{x+1}+\frac{2}{x}=3$; to find x.

$$6+\frac{2x+2}{x}=3x+3,$$

$$6x+2x+2=3x^2+3x.$$

$$\therefore\ 3x^2-5x=2.$$

$$x^2-\tfrac{5}{3}x=\tfrac{2}{3}$$

$$x^2-\tfrac{5}{3}x+(\tfrac{5}{6})^2=\tfrac{25}{36}+\tfrac{2}{3}=\tfrac{49}{36}.$$

$$x-\tfrac{5}{6}=\pm\tfrac{7}{6}$$

$$\therefore\ x=\tfrac{5}{6}\pm\tfrac{7}{6}=2 \text{ or } -\tfrac{1}{3}.$$

186. *Ex.* 4. Let $x+\sqrt{(5x+10)}=8$, to find x.

By transp. $\sqrt{(5x+10)}=8-x$.

Squaring $5x+10=64-16x+x^2$.

$\therefore\ x^2-21x=-54$.

$\therefore\ x^2-21x+(\frac{21}{2})^2=-54+\frac{441}{4}=\frac{225}{4}$.

$\therefore\ x-\frac{21}{2}=\pm\frac{15}{2}$.

$\therefore\ x=\frac{21}{2}\pm\frac{15}{2}=18$ or 3.

By this process two values of x are found; but on trial it appears, that 18 does not answer the conditions of the equation, if we suppose that $\sqrt{(5x+10)}$ represents the positive square root of $5x+10$. The reason is, that $5x+10$ is the square of $-\sqrt{(5x+10)}$ as well as of $+\sqrt{(5x+10)}$; thus by squaring both sides of the equation $\sqrt{(5x+10)}=8-x$, a new condition is introduced, and a new value of the unknown quantity corresponding to it, which had no place before. Here 18 is the value which corresponds to the supposition that, $x-\sqrt{(5x+10)}=8$.

It should be particularly observed, that since $(+x)\times(+y)$ is equal to $(-x)\times(-y)$, in the multiplication and involution of quantities new values are always introduced, which, if not again excluded by the nature of the question, will appear in the final equation.

Ex. 5. $\dfrac{\sqrt{(x^2+4)}+\sqrt{(x+1)}}{\sqrt{(x^2+4)}-\sqrt{(x+1)}}=3$; to find x.

$\therefore\ \dfrac{\sqrt{(x^2+4)}}{\sqrt{(x+1)}}=\dfrac{3+1}{3-1}=\frac{4}{2}=2.$ Art. 148.

$\sqrt{(x^2+4)}=2\sqrt{(x+1)}$.

$\therefore\ x^2+4=4x+4$.

$\therefore\ x^2-4x=0$

$\therefore x(x-4)=0$; $\therefore\ x=0$ or $x-4=0$.

or $x^2-4x+4=4$

$\therefore\ x-2=\pm 2$

$x=2\pm 2=4$ or 0.

187. An equation of the form $ax^2+bx+c=0$, (where a, b, c, are any quantities whatever) may, however, be solved as follows, without dividing by the coe-fficient of x^2.

Multiply the whole equation by $4a$, that is, four times the coe-fficient of x^2, and add b^2 (the square of the coe-fficient of x) to each side; then

$$4a^2x^2+4abx+b^2=b^2-4ac;$$

$$\therefore\ 2ax+b=\pm\sqrt{(b^2-4ac)}$$

$$\therefore\ x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}.$$

Ex. $8x^2-13x+5=0$; to find x.

Multiply by 4×8, then

$$256x^2-416x=-160.$$

Add $(13)^2$ to each side, then

$$256x^2-416x+169=-160+169=9.$$

$$\therefore\ 16x-13=\pm 3.$$

$$\therefore\ x=\frac{13\pm 3}{16}=1 \text{ or } \tfrac{5}{8}.$$

This method of solving quadratic equations may be found in the *Vija Ganita* of Bhaskar Achárjia, a famous mathematician who lived about the beginning of the thirteenth century.

Ex. 6. $ax(ax-2)+4x(a+x)+a^2+1=(a+2x)^2$; to find x.

$$a^2x^2-2ax+4ax+4x^2+a^2+1=a^2+4ax+4x^2,$$

$$a^2x^2-2ax+1=0.$$

$$\therefore\ x^2-\frac{2}{a}x=-\frac{1}{a^2}$$

$$\therefore\ x^2-\frac{2}{a}x+\frac{1}{a^2}=-\frac{1}{a^2}+\frac{1}{a^2}=0.$$

$$\therefore\ x-\frac{1}{a}=0;$$

$$\therefore\ x=\frac{1}{a}.$$

Here the two values of x are equal or are the same.

Acharjia's method. $a^2x^2-2ax=-1.$

Multiply by $4a^2$, $4a^4x^2-8a^3x=-4a^2$

Add $4a^2$, $4a^4x^2-8a^3x+4a^2=-4a^2+4a^2=0$

$$\therefore 2a^2x-2a=0$$

$$\therefore ax-1=0, \text{ etc.}$$

Ex. 7. Let $x^2+4x+25=0$; to find x.

$$x^2+4x+4=-25+4=-21.$$

$$\therefore x+2=\pm\sqrt{(-21)}$$

$$\therefore x=-2\pm\sqrt{(-21)}.$$

To extract the square root of -21 is impossible, therefore the values of x are *impossible* or *imaginary*.

Ex. 8. Let $5x^2-21x-20=0$, to find the values of x.

$$x^2-\tfrac{21}{5}x=4.$$

$$x^2-\tfrac{21}{5}x+\tfrac{441}{100}=4+\tfrac{441}{100}=\tfrac{841}{100}=(\tfrac{29}{10})^2.$$

$$\therefore x-\tfrac{21}{10}=\pm\tfrac{29}{10};$$

$$\therefore x=\tfrac{21}{10}\pm\tfrac{29}{10}=5 \text{ or } -\tfrac{4}{5}.$$

Ex. 9. $x^2-100x+196=0$, to find x.

$$x^2-100x+50^2=50^2-196$$

$$\therefore x-50=\pm\sqrt{\{50^2-14^2\}}$$

$$=\pm\sqrt{\{(50+14)(50-14)\}}$$

$$=\pm\sqrt{(64\times 36)}$$

$$=\pm(8\times 6)=\pm 48$$

$$\therefore x=50\pm 48=98 \text{ or } 2.$$

Ex. 10. Let $adx-acx^2=bcx-bd$; to find x.

$$acx^2-adx+bcx=bd.$$

$$x^2-\frac{ad-bc}{ac}x=\frac{bd}{ac}.$$

$$x^2-\frac{ad-bc}{ac}x+\left(\frac{ad-bc}{2ac}\right)^2=\frac{bd}{ac}+\left(\frac{ad-bc}{2ac}\right)^2=\left(\frac{ad+bc}{2ac}\right)^2$$

$$\therefore x-\frac{ad-bc}{2ac}=\pm\frac{ad+bc}{2ac}.$$

$$\therefore\ x=\frac{ad-bc}{2ac}\pm\frac{ad+bc}{2ac}=\frac{2ad}{2ac}\text{ or }\frac{-2bc}{2ac}$$

$$\therefore\ x=\frac{d}{c}\text{ or }-\frac{b}{a}.$$

Ex. 11. $256x^2-40x+1=0$; to find x.

$$x^2-\tfrac{5}{32}x=-\tfrac{1}{256}$$

$$x^2-\tfrac{5}{32}x+(\tfrac{5}{64})^2=(\tfrac{5}{64})^2-(\tfrac{1}{16})^2$$

$$=(\tfrac{5}{64}+\tfrac{1}{16})(\tfrac{5}{64}-\tfrac{1}{16})$$

$$=\tfrac{9}{64}\times\tfrac{1}{64}=(\tfrac{3}{64})^2.$$

$$\therefore\ x-\tfrac{5}{64}=\pm\tfrac{3}{64}.$$

$$\therefore\ x=\tfrac{5}{64}\pm\tfrac{3}{64}=\tfrac{1}{8}\text{ or }\tfrac{1}{32}.$$

Ex. 12. $(x-\frac{1}{2})(x-\frac{1}{3})+(x-\frac{1}{3})(x-\frac{1}{4})=(2x-\frac{3}{4})(2x-\frac{1}{4})$; to find x.

$$(x-\tfrac{1}{3})\{x-\tfrac{1}{2}+x-\tfrac{1}{4}\}=(2x-\tfrac{3}{4})(2x-\tfrac{1}{4}).$$

$$\therefore\ (2x-\tfrac{3}{4})(x-\tfrac{1}{3})=(2x-\tfrac{3}{4})(2x-\tfrac{1}{4}).$$

$$(2x-\tfrac{3}{4})(2x-\tfrac{1}{4}-x+\tfrac{1}{3})=0$$

$$(2x-\tfrac{3}{4})(x+\tfrac{1}{12})=0.$$

$$\therefore\ 2x-\tfrac{3}{4}=0,\text{ or }x+\tfrac{1}{12}=0.$$

$$\therefore\ 2x=\tfrac{3}{4}\text{ or }x=-\tfrac{1}{12}.$$

$$\therefore\ x=\tfrac{3}{8}\text{ or }x=-\tfrac{1}{12}.$$

Ex. 13. $\dfrac{x-2}{x-4}+\dfrac{x-3}{x-2}=\dfrac{x-2}{x-1}+\dfrac{x-1}{x-3}$; to find x.

$$\frac{x-2}{x-4}-\frac{x-1}{x-3}=\frac{x-2}{x-1}-\frac{x-3}{x-2}.$$

$$\therefore\ \frac{x^2-5x+6-x^2+5x-4}{(x-4)(x-3)}=\frac{x^2-4x+4-x^2+4x-3}{(x-1)(x-2)}$$

$$\therefore\ \frac{2}{(x-4)(x-3)}=\frac{1}{(x-1)(x-2)}$$

$$\therefore\ 2(x-1)(x-2)=(x-4)(x-3).$$

$$2x^2-6x+4=x^2-7x+12.$$

$$x^2+x=8$$

$$x^2+x+\tfrac{1}{4}=8+\tfrac{1}{4}=\tfrac{33}{4}$$

$$x+\tfrac{1}{2}=\pm\tfrac{1}{2}\sqrt{(33)}$$

$$\therefore\ x=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{(33)}.$$

Ex. 14. $(4-2\sqrt{3})x^2-(\sqrt{3}-1)x=2$: to find x.

$$x^2-\frac{\sqrt{3}-1}{(\sqrt{3}-1)^2}x=\frac{2}{(\sqrt{3}-1)^2}.$$

$$\therefore\ x^2-\tfrac{1}{2}(\sqrt{3}+1)x=\tfrac{1}{2}(\sqrt{3}+1)^2.$$

$$\therefore\ x^2-\tfrac{1}{2}(\sqrt{3}+1)x+\tfrac{1}{16}(\sqrt{3}+1)^2$$
$$=\tfrac{1}{2}(\sqrt{3}+1)^2+\tfrac{1}{16}(\sqrt{3}+1)^2=\tfrac{9}{16}(\sqrt{3}+1)^2.$$

$$\therefore\ x-\tfrac{1}{4}(\sqrt{3}+1)=\pm\tfrac{3}{4}(\sqrt{3}+1).$$

$$\therefore\ x=\tfrac{1}{4}(\sqrt{3}+1)\pm\tfrac{3}{4}(\sqrt{3}+1)=\sqrt{3}+1 \text{ or } -\tfrac{1}{2}(\sqrt{3}+1).$$

Ex. 15. $\dfrac{1}{x+a}+\dfrac{1}{x+b}-\dfrac{1}{x-c}=0$; to find x.

Multiply by $(x+a)(x+b)(x-c)$, then

$$(x+b)(x-c)+(x+a)(x-c)-(x+a)(x+b)=0.$$

$$\therefore\ x^2-2cx=ac+bc+ab.$$

$$x^2-2cx+c^2=c^2+ac+bc+ab.$$

$$x-c=\pm\sqrt{\{(b+c)(a+c)\}};$$

$$\therefore\ x=c\pm\sqrt{\{(b+c)(a+c)\}}.$$

Ex. 16. $\dfrac{x^2+1}{a^2+3ab+b^2}=\dfrac{2x}{a^2+ab+b^2}$, to find x.

$$\frac{x^2+1}{2x}=\frac{a^2+3ab+b^2}{a^2+ab+b^2}.$$

$$\frac{x^2+2x+1}{x^2-2x+1}=\frac{a^2+2ab+b^2}{ab}. \qquad \text{Art. 148.}$$

$$\frac{x+1}{x-1}=\pm\frac{a+b}{\sqrt{(ab)}}=\frac{a+b}{\pm\sqrt{(ab)}}$$

$$\therefore\ x=\frac{a\pm\sqrt{(ab)}+b}{a\mp\sqrt{(ab)}+b}.$$

Ex. 17. $2x+\sqrt{\{4x^2+\sqrt{(7x^2+2)}\}}=1$; to find x.

$$\sqrt{\{4x^2+\sqrt{(7x^2+2)}\}}=1-2x.$$

$$4x^2+\sqrt{(7x^2+2)}=1-4x+4x^2.$$

$$\therefore\ \sqrt{(7x^2+2)}=1-4x.$$

$$\therefore\ 7x^2+2=1-8x+16x^2.$$

$$\therefore\ 9x^2-8x=1.$$

$$\therefore\ x^2 - \tfrac{8}{9}x + \left(\tfrac{4}{9}\right)^2 = \tfrac{1}{9} + \left(\tfrac{4}{9}\right)^2 = \tfrac{25}{81},$$

$$\therefore\ x - \tfrac{4}{9} = \pm\tfrac{5}{9}.$$

$$\therefore\ x = \tfrac{4}{9} \pm \tfrac{5}{9} = 1 \text{ or } -\tfrac{1}{9}.$$

Ex. 18. $\dfrac{x-5}{\sqrt{(x-1)}+2} + \dfrac{x-3}{\sqrt{(x-2)}-1} = \dfrac{3x-6}{\sqrt{(3x-5)}+1}$; to find x.

$$\frac{(x-1)-4}{\sqrt{(x-1)}+2} + \frac{(x-2)-1}{\sqrt{(x-2)}-1} = \frac{(3x-5)-1}{\sqrt{(3x-5)}+1}.$$

$$\therefore\ \sqrt{(x-1)} - 2 + \sqrt{(x-2)} + 1 = \sqrt{(3x-5)} - 1$$

$$\therefore\ \sqrt{(x-1)} + \sqrt{(x-2)} = \sqrt{(3x-5)}.$$

$$\therefore\ x - 1 + x - 2 + 2\sqrt{(x^2-3x+2)} = 3x - 5.$$

$$\therefore\ 2\sqrt{(x^2-3x+2)} = x - 2.$$

$$4x^2 - 12x + 8 = x^2 - 4x + 4$$

$$3x^2 - 8x = -4$$

$$x^2 - \tfrac{8}{3}x = -\tfrac{4}{3}$$

$$x^2 - \tfrac{8}{3}x + \left(\tfrac{4}{3}\right)^2 = -\tfrac{4}{3} + \left(\tfrac{4}{3}\right)^2 = \tfrac{4}{9}.$$

$$x - \tfrac{4}{3} = \pm\tfrac{2}{3}.$$

$$\therefore\ x = \tfrac{4}{3} \pm \tfrac{2}{3} = 2 \text{ or } \tfrac{2}{3}.$$

Ex. 19. $(1-x+x^2)^2 = \dfrac{1-a}{1+a}(1+x^2+x^4)$; to find x.

$$1 - x + x^2 = \frac{1-a}{1+a} \times \frac{1+x^2+x^4}{1-x+x^2} = \frac{1-a}{1+a}(1+x+x^2).$$

$$\therefore\ \frac{1-x+x^2}{1+x+x^2} = \frac{1-a}{1+a}$$

$$\therefore\ \frac{1+x^2}{-x} = \frac{1}{-a}; \qquad \therefore\ \frac{1+x^2}{x} = \frac{1}{a}$$

$$\therefore\ a + ax^2 = x; \qquad \therefore\ ax^2 - x = -a.$$

$$\therefore\ x^2 - \frac{x}{a} + \left(\frac{1}{2a}\right)^2 = \frac{1}{4a^2} - 1.$$

$$\therefore\ x - \frac{1}{2a} = \pm\sqrt{\left(\frac{1}{4a^2} - 1\right)}.$$

$$x = \frac{1}{2a}\left\{1 \pm \sqrt{(1-4a^2)}\right\}.$$

Ex. 20. $\sqrt{(x^2-3x+2)}-\sqrt{(x^2-3x-6)}$
$=\sqrt{(20)}-\sqrt{(12)}$, to find x.

$$(x^2-3x+2)-(x^2-3x-6)=8=20-12.$$

By division $\sqrt{(x^2-3x+2)}+\sqrt{(x^2-3x-6)}=\sqrt{(20)}+\sqrt{(12)}$

By addition $2\sqrt{(x^2-3x+2)}=2\sqrt{(20)}$.

$$x^2-3x+2=20.$$
$$\therefore\ x^2-3x+\tfrac{9}{4}=18+\tfrac{9}{4}=\tfrac{81}{4}$$
$$x-\tfrac{3}{2}=\pm\tfrac{9}{2}.$$
$$\therefore\ x=\tfrac{3}{2}\pm\tfrac{9}{2}=6 \text{ or } -3.$$

Ex. 21. $a+x-\sqrt{(3a^2+2ax-x^2)}$
$=\sqrt{(3a^2-4ax+x^2)}+\sqrt{(a^2-x^2)}$; to find x.

$$a+x-\sqrt{(a^2-x^2)}=\sqrt{\{(3a-x)(a-x)\}}+\sqrt{\{(3a-x)(a+x)\}}.$$
$$\sqrt{(a+x)}\{\sqrt{(a+x)}-\sqrt{(a-x)}\}=\sqrt{(3a-x)}\{\sqrt{(a-x)}+\sqrt{(a+x)}\}.$$
$$\frac{\sqrt{(a+x)}}{\sqrt{(3a-x)}}=\frac{\sqrt{(a-x)}+\sqrt{(a+x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}$$
$$\therefore\ \frac{a+x}{3a-x}=\frac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}.$$
$$\frac{2a}{x-a}=\frac{a}{\sqrt{(a^2-x^2)}} \qquad \text{Art. 148.}$$
$$2\sqrt{(a^2-x^2)}=x-a$$
$$4a^2-4x^2=x^2-2ax+a^2$$
$$5x^2-2ax=3a^2$$
$$x^2-\frac{2}{5}ax+\left(\frac{1}{5}a\right)^2=\frac{1}{25}a^2+\frac{3}{5}a^2=\frac{16}{25}a^2$$
$$\therefore\ x-\tfrac{1}{5}a=\pm\tfrac{4}{5}a.$$
$$\therefore\ x=\tfrac{1}{5}a\pm\tfrac{4}{5}a=a \text{ or } -\tfrac{3}{5}a.$$

Ex. 22. To Solve

$$\left(\frac{8x-1-4\sqrt{(4x^2-x)}}{8x-1+4\sqrt{(4x^2-x)}}\right)^{\frac{m}{2}}=(4x^2-x)^{\frac{2m+1}{4}}\left(\frac{2\sqrt{x}+\sqrt{(4x-1)}}{2\sqrt{x}-\sqrt{(4x-1)}}\right)^{\frac{1}{2}}$$
$$\pm\left(\frac{2\sqrt{x}-\sqrt{(4x-1)}}{2\sqrt{x}+\sqrt{(4x-1)}}\right)^{m}=(4x^2-x)^{\frac{2m+1}{4}}\left(\frac{2\sqrt{x}+\sqrt{(4x-1)}}{2\sqrt{x}-\sqrt{(4x-1)}}\right)^{\frac{1}{2}};$$
$$\therefore\ \pm\left(\frac{2\sqrt{x}-\sqrt{(4x-1)}}{2\sqrt{x}+\sqrt{(4x-1)}}\right)^{\frac{2m+1}{2}}=(4x^2-x)^{\frac{2m+1}{4}};$$

$$\left(\frac{2\sqrt{x}-\sqrt{(4x-1)}}{2\sqrt{x}+\sqrt{(4x-1)}}\right)^{2m+1}=(4x^2-x)^{\frac{2m+1}{2}}.$$

Extracting $2m+1th$ root of both the sides and rationalizing the denominator of the left-hand member, we have

$$\{2\sqrt{x}-\sqrt{(4x-1)}\}^2=\sqrt{(4x^2-x)}.$$

$$\therefore\ 4x+4x-1-4\sqrt{(4x^2-x)}=\sqrt{(4x^2-x)}.$$

$$\therefore\ 8x-1=5\sqrt{(4x^2-x)}.$$

$$64x^2-16x+1=100x^2-25x.$$

$$\therefore\ 36x^2-9x=1.$$

$$\therefore\ x^2-\tfrac{1}{4}x+\tfrac{1}{64}=\tfrac{1}{36}+\tfrac{1}{64}=\tfrac{25}{576}.$$

$$\therefore\ x-\tfrac{1}{8}=\pm\tfrac{5}{24}.$$

$$\therefore\ x=\tfrac{1}{8}\pm\tfrac{5}{24}=\tfrac{1}{3}\text{ or }-\tfrac{1}{12}.$$

EXAMPLES LXVI.

Solve the following equations,

1. $x^2-3x+2=0.$
2. $x^2-5x+6=0.$
3. $x^2-7x+6=0.$
4. $2x^2-5x+2=0.$
5. $6x^2-5x+1=0.$
6. $6x^2-13x+6=0.$
7. $2x^2+5x-3=0.$
8. $6x^2+13x+6=0.$
9. $6x^2+x-2=0.$
10. $9x^2-9x-4=0.$
11. $3x^2-2x=133.$
12. $3x^2+2x-85=0.$
13. $3x^2-2x-65=0.$
14. $\sqrt{(x^2-9)}=2x-6.$
15. $9x^2+9x-4=0.$
16. $4x-3x^2=6x-8.$
17. $x(2x+1)=(x+3)(x+8).$
18. $2x^2-2(x-1)^2=3x(x-1).$
19. $x^2+(x-7)^2=\frac{29}{10}(7x-x^2).$
20. $x^2-(a-b)x=ab.$
21. $x^2-2ax=b^2-a^2.$
22. $x^2+100x=1100.$
23. $(2x-3)^2-40=(x-2)^2.$
24. $(3x-2)^2=4(2x-3)^2.$
25. $(3x-1)^2-6(3x-1)+(2x-3)^2=25.$
26. $(x-\frac{1}{2})(x-\frac{1}{3})+(x-\frac{1}{2})(x-\frac{1}{4})=(2x-\frac{7}{12})(2x-\frac{1}{2})-6\frac{5}{6}.$

27. $\dfrac{x^2+1}{a^2+2a+2}=\dfrac{x}{a+1}$.

28. $\dfrac{2x}{x-4}+\dfrac{2x-5}{x-3}=8\frac{1}{3}$.

29. $\dfrac{x+5}{x+9}+\dfrac{x+3}{x+7}=1\dfrac{19}{99}$.

30. $9x+\dfrac{1}{x}=\dfrac{29}{x}+4$.

31. $\dfrac{x}{2}+\dfrac{2}{x}=\dfrac{x}{3}+\dfrac{4}{3}$.

32. $\dfrac{x+1}{x+2}+\dfrac{x-1}{x-2}=\dfrac{4x+2}{2x-1}$.

33. $\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}=\dfrac{4x-7}{2x-4}$.

34. $\dfrac{x^2}{a+b}-\dfrac{x}{a-b}+\dfrac{2b}{a^2-b^2}=0$.

35. $\dfrac{x+3}{x+5}-\dfrac{x+1}{x+3}=\dfrac{3x-5}{3x-7}-\dfrac{3x-3}{3x-5}$.

36. $\dfrac{x-3}{x-5}-\dfrac{x-1}{x-3}=\dfrac{3x+5}{3x+7}-\dfrac{3x+3}{3x+5}$.

37. $\dfrac{x+2}{x+1}-\dfrac{x+3}{x+2}=\dfrac{4x-7}{4x-5}-\dfrac{4x-9}{4x-7}$.

38. $x^2-(a+b)x+ab=0$.

39. $x^2-(a-b)x=ab$.

40. $x^2+a^5=(a+1)a^2x$.

41. $x^2-2ax+a^2=n^2$.

42. $a^2x^2-3ax+2=0$.

43. $acx^2=(bc+ab)x-b^2$.

44. $x^2-ax+\frac{1}{2}(a+x-b)b=0$.

45. $(x^2+1)(a^2+ab+b^2)=2x(a^2+3ab+b^2)$.

46. $4b^2x-(b^2-a^2+x)^2=0$.

47. $(b-a)x^2+2a=(a+b)x$.

48. $x^2-ax+\frac{1}{2}a+\frac{1}{2}(x-1)=0$.

49. $\dfrac{1}{x}+\dfrac{1}{x+a}=\dfrac{1}{b}+\dfrac{1}{a+b}$.

50. $\dfrac{x^2}{a^2}=\dfrac{2x+1}{1-a^2}$.

51. $a^2x^2+b^2+c^2=a^4+2bc+2ax(b-c)$.

52. $\dfrac{1}{(a+b)(a+c)}+\dfrac{a-c}{(x-a)(x-b)(a+b)}=\dfrac{1}{(x-a)(a+c)}$.

53. $\dfrac{1}{ab-ax}+\dfrac{1}{bc-bx}=\dfrac{x-a-c}{a(b-x)(c-x)}+\dfrac{x}{ab(b-x)}$.

54. $\dfrac{(a^2+b^2)x-2(a-b)^2+4b^2}{(a^2+b^2)x-2(a+b)^2+4b^2}=\dfrac{(x+1)^2-3}{(x-1)^2-3}$.

55. $x^2 - 3(2+\sqrt{3})x + 2(7+4\sqrt{3}) = 0.$

56. $(11-6\sqrt{2})x^2 + (3-\sqrt{2})x - 6 = 0.$ (Answer)?

57. $(x^2+5)(x-\sqrt{3}) = (x+\sqrt{3})(x^2-3).$

58. $\dfrac{\sqrt{x}+1}{\sqrt{x}-1} + 5\dfrac{\sqrt{x}-1}{\sqrt{x}+1} = \dfrac{14}{3}.$

59. $\dfrac{c}{a-c}\left(x+\dfrac{1}{x}\right) = 1 + \dfrac{a+c}{(a-c)x} + \dfrac{b}{(a-c)}\left(1+\dfrac{1}{x}\right).$

60. $x^2 + \dfrac{\sqrt{3}-1}{\sqrt{3}+1}x + \dfrac{2-\sqrt{3}}{2+\sqrt{3}} = 15.$

61. $\dfrac{3}{1+3\sqrt{x}} + \dfrac{2}{\sqrt{x}(2x-1)} = \dfrac{2\sqrt{x}}{2x-1}.$

62. $\dfrac{\sqrt{x}}{\sqrt{x}-\sqrt{a}} - \dfrac{\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \dfrac{x-a}{x+a} + 2n.$

63. $\sqrt{(2x+9)} - \sqrt{(x-4)} = \sqrt{(x+1)}.$

64. $\sqrt{(x+7a)} - \sqrt{(x+a)} = \sqrt{(2x)}.$

65. $\sqrt{(x+2)} + \sqrt{(x-1)} = \sqrt{(2x+5)}.$

66. $\sqrt{(x+1)} + \sqrt{(x-2)} = \sqrt{(x+6)}.$

67. $\sqrt{(a^2+bx)} + \sqrt{(a^2-bx)} = \sqrt{(bx+a^2)}.$

68. $\sqrt{(x^2+3x+9)} - \sqrt{(x^2-9)} = 3.$

69. $\sqrt{(2x+1)} + \sqrt{(7x-27)} = \sqrt{(3x+4)}.$

70. $\sqrt{(2x+7)} + \sqrt{(3x-18)} = \sqrt{(7x+1)}.$

71. $\sqrt{(2x-1)} + \sqrt{(3x+10)} = \sqrt{(13x-1)}.$

72. $\sqrt{(5-x)} + \sqrt{(5+x)} = \dfrac{12}{\sqrt{(5+x)}}.$

73. $\dfrac{\sqrt{(12-x)}}{5} = \dfrac{3}{\sqrt{(12-x)}+2}.$

74. $\dfrac{1+\sqrt{(1-2x)}}{1-\sqrt{(1-2x)}} + \dfrac{2-\sqrt{(1-2x)}}{x} = x - \tfrac{4}{3}.$

75. $\dfrac{x+\sqrt{(12a-x)}}{x-\sqrt{(12a-x)}} = \dfrac{\sqrt{a}+1}{\sqrt{a}-1}.$

76. $\dfrac{1}{\sqrt{(x+a)}}+\dfrac{1}{\sqrt{(x-a)}}=\dfrac{\sqrt{(x+2a)}}{\sqrt{(x^2-a^2)}}.$

77. $\dfrac{\sqrt{(2a^2-x^2)}+b\sqrt{(2a-x)}}{\sqrt{(2a^2-x^2)}-b\sqrt{(2a-x)}}=\dfrac{\sqrt{a}+b}{\sqrt{a}-b}.$

78. $\dfrac{x-3}{\sqrt{(x+1)}-2}+\dfrac{x-2}{\sqrt{(x-1)}+1}=\dfrac{5(x-1)}{\sqrt{(5x-4)}-1}.$

79. $\dfrac{x-2a}{\sqrt{(x-a)}+\sqrt{a}}+\dfrac{x-3a}{\sqrt{(x+a)}-2\sqrt{a}}=\dfrac{2x-5a}{\sqrt{(2x-a)}-2\sqrt{a}}.$

80. $x^2+ax+\frac{1}{4}a^2=\dfrac{n+1}{n-1}\sqrt{(x^4-\frac{1}{2}a^2x^2+\frac{1}{16}a^4)}.$

81. $a-x+\sqrt{(3a^2-2ax-x^2)}=\sqrt{(3a^2+4ax+x^2)}-\sqrt{(a^2-x^2)}.$

82. $\sqrt{(2x^2+x-1)}+\sqrt{(3x-1-2x^2)}$
$$=\sqrt{(3+x-2x^2)}-\sqrt{(2x^2-5x+3)}.$$

83. $\sqrt{(ax+2x^2)}+\sqrt{(2a^2-5ax+2x^2)}.$
$$=\sqrt{(2a^2+3ax-2x^2)}-\sqrt{(ax-2x^2)}.$$

84. $\left(\dfrac{a+\sqrt{(2ax-x^2)}}{a-\sqrt{(2ax-x^2)}}\right)^m=\left(\dfrac{2a-x}{x}\right)^{\frac{2m^2+1}{2m}}\left(\dfrac{\sqrt{(2a-x)}-\sqrt{x}}{\sqrt{(2a-x)}+\sqrt{x}}\right)^{\frac{1}{m}}$

Equations which may be Solved like Quadratics.

188. Every equation of the form $x^{2n}\pm qx^n=p$, where the unknown quantity is only found in two terms, and its index in one is twice as great as in the other, may be solved by processes similar to those given in the preceding chapter.

Ex. 1. Let $x^4-13x^2+36=0$, to find x.

By transp. $x^4-13x^2=-36.$

$$x^4-13x^2+(\tfrac{13}{2})^2=(\tfrac{13}{2})^2-36.$$
$$=(\tfrac{13}{2}+6)(\tfrac{13}{2}-6).$$
$$=\tfrac{25}{2}\times\tfrac{1}{2}=(\tfrac{5}{2})^2.$$
$$x^2-\tfrac{13}{2}=\pm\tfrac{5}{2}.$$
$$\therefore x^2=\tfrac{13}{2}\pm\tfrac{5}{2}=9 \text{ or } 4.$$
$$\therefore x=\pm 3 \text{ or } \pm 2.$$

Ex. 2. Let $x^4-6x^2-27=0$, to find x.

$$x^4-6x^2=27.$$
$$x^4-6x^2+9=27+9=36.$$
$$x^2-3=\pm 6$$
$$\therefore\ x^2=3\pm 6=9 \text{ or } -3$$
$$\therefore\ x=\pm 3 \text{ or } \pm\sqrt{(-3)}.$$

Two of the values of x, $+3$ and -3, are *real* and the other two values are *imaginary*.

Ex. 3. Let $x+4x^{\frac{1}{2}}=21$, to find x.

$$x+4x^{\frac{1}{2}}+4=21+4=5^2.$$
$$x^{\frac{1}{2}}+2=\pm 5.$$
$$x^{\frac{1}{2}}=-2\pm 5=3 \text{ or } -7$$
$$\therefore\ x=9 \text{ or } 49.$$

Ex. 4. Let $x^{-1}+x^{-\frac{1}{2}}=6$, to find x.

$$x^{-1}+x^{-\frac{1}{2}}+\tfrac{1}{4}=6+\tfrac{1}{4}=\tfrac{25}{4}.$$
$$\therefore\ x^{-\frac{1}{2}}+\tfrac{1}{2}=\pm\tfrac{5}{2}$$
$$\therefore\ x^{-\frac{1}{2}}=-\tfrac{1}{2}\pm\tfrac{5}{2}=2 \text{ or } -3.$$
$$x^{\frac{1}{2}}=\tfrac{1}{2} \text{ or } -\tfrac{1}{3}.$$
$$\therefore\ x=\tfrac{1}{4} \text{ or } \tfrac{1}{9}.$$

Ex. 5. Let $x^{2n}-(a+b)x^n+ab=0$, to find x.

$$x^{2n}-(a+b)x^n+\{\tfrac{1}{2}(a+b)\}^2=\{\tfrac{1}{2}(a+b)\}^2-ab$$
$$=\{\tfrac{1}{2}(a-b)\}^2.$$
$$\therefore\ x^n-\tfrac{1}{2}(a+b)=\pm\tfrac{1}{2}(a-b).$$
$$\therefore\ x^n=\tfrac{1}{2}\{a+b\pm(a-b)\}$$
$$=a \text{ or } b.$$

By extracting the nth root, the values of x are known.

Ex. 6. Let $x^{\frac{2}{3}}-5x^{\frac{1}{3}}+6=0$, to find x.

$$x^{\frac{2}{3}}-5x^{\frac{1}{3}}=-6$$

$$\therefore\ x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + \tfrac{25}{4} = \tfrac{25}{4} - 6 = \tfrac{1}{4}$$

$$\therefore\ x^{\frac{1}{3}} - \tfrac{5}{2} = \pm\tfrac{1}{2}.$$

$$\therefore\ x^{\frac{1}{3}} = \tfrac{5}{2} \pm \tfrac{1}{2} = 3 \text{ or } 2.$$

$$\therefore\ x = 27 \text{ or } 8.$$

Ex. 7. $x^n\sqrt{x^n} - \dfrac{9x^n}{\sqrt[4]{x^n}} + 8 = 0$, to find x.

$$x^{n+\frac{n}{2}} - 9x^{n-\frac{n}{4}} + 8 = 0.$$

$$\therefore\ x^{\frac{3n}{2}} - 9x^{\frac{3n}{4}} + 8 = 0.$$

$$x^{\frac{3n}{2}} - 9x^{\frac{3n}{4}} + (\tfrac{9}{2})^2 = (\tfrac{9}{2})^2 - 8 = \tfrac{49}{4}.$$

$$x^{\frac{3n}{4}} - \tfrac{9}{2} = \pm\tfrac{7}{2}$$

$$\therefore\ x^{\frac{3n}{4}} = \tfrac{9}{2} \pm \tfrac{7}{2} = 8 \text{ or } 1$$

Extracting the cube root of both the sides.

$$x^{\frac{n}{4}} = 2 \text{ or } 1.$$

$$\therefore\ x^n = 16 \text{ or } 1.$$

By extracting the n^{th} root the values of x are known.

Ex. 8. Let $\sqrt[n]{2}(x^{\frac{1}{n}} + x^{-\frac{1}{n}}) = 4^{\frac{1}{n}} + 1$, to find x.

Multiply by $\dfrac{\sqrt[n]{x}}{\sqrt[n]{2}}$, $x^{\frac{2}{n}} + 1 = \left(\dfrac{\sqrt[n]{4}+1}{\sqrt[n]{2}}\right)x^{\frac{1}{n}}$.

$$\therefore\ x^{\frac{2}{n}} - \left(\frac{\sqrt[n]{4}+1}{\sqrt[n]{2}}\right)x^{\frac{1}{n}} + \left(\frac{\sqrt[n]{4}+1}{2\sqrt[n]{2}}\right)^2 = \left(\frac{\sqrt[n]{4}+1}{2\sqrt[n]{2}}\right)^2 - 1$$

$$= \frac{(\sqrt[n]{4})^2 + 2\sqrt[n]{4} + 1 - 4\sqrt[n]{4}}{4\sqrt[n]{4}}$$

$$= \left(\frac{\sqrt[n]{4}-1}{2\sqrt[n]{2}}\right)^2$$

$$\therefore\ x^{\frac{1}{n}} - \frac{\sqrt[n]{4}+1}{2\sqrt[n]{2}} = \pm\frac{\sqrt[n]{4}-1}{2\sqrt[n]{2}}.$$

$$\therefore\ x^{\frac{1}{n}} = \frac{\sqrt[n]{4}+1}{2\sqrt[n]{2}} \pm \frac{\sqrt[n]{4}-1}{2\sqrt[n]{2}} = \sqrt[n]{2} \text{ or } \frac{1}{\sqrt[n]{2}}.$$

$$\therefore\ x = 2 \text{ or } \tfrac{1}{2}.$$

Ex. 9. Let $(x^{2^{n-1}}+a^{2^{n-1}})(x^{2^{n-1}}-3a^{2^{n-1}})$

$$=a^{2^{n-1}}(x^{2^{n-1}}-5a^{2^{n-1}}), \text{ to find } x.$$

Because $(x^{2^{n-1}})^2=x^{2\times 2^{n-1}}=x^{2^n}$.

$$\therefore\ x^{2^n}-2a^{2^{n-1}}x^{2^{n-1}}-3a^{2^n}=a^{2^{n-1}}x^{2^{n-1}}-5a^{2^n}.$$

$$\therefore\ x^{2^n}-3a^{2^{n-1}}x^{2^{n-1}}+\tfrac{9}{4}a^{2^n}=\tfrac{9}{4}a^{2^n}-2a^{2^n}=\tfrac{1}{4}a^{2^n}.$$

$$x^{2^{n-1}}-\tfrac{3}{2}a^{2^{n-1}}=\pm\tfrac{1}{2}a^{2^{n-1}}.$$

$$\therefore\ x^{2^{n-1}}=\tfrac{3}{2}a^{2^{n-1}}\pm\tfrac{1}{2}a^{2^{n-1}}$$

$$=2a^{2^{n-1}} \text{ or } a^{2^{n-1}}.$$

Extracting $2^{n-1}th$ root the values of x are found.

189. The equations, which are reducible to the form $X^{2n}\pm pX^n\pm q=0$, (in which X represents any simple or quadratic expression involving the unknown quantity) may also be solved like quadratics.

Ex. 10. Let $x^4-10x^3+35x^2-50x+24=0$, to find the values of x.

$$(x^2-5x)^2+10(x^2-5x)+24=0.$$

Let $x^2-5x=y$, then

$$y^2+10y+24=0.$$

$$\therefore\ y^2+10y+25=25-24=1.$$

$$\therefore\ y+5=\pm 1.$$

$$\therefore\ y=-4 \text{ or } -6.$$

$$\therefore\ x^2-5x=-4 \text{ or } -6.$$

$$\therefore\ x^2-5x+\tfrac{25}{4}=\tfrac{25}{4}-4 \text{ or } \tfrac{25}{4}-6$$

$$=\tfrac{9}{4} \text{ or } \tfrac{1}{4}.$$

$$\therefore\ x-\tfrac{5}{2}=\pm\tfrac{3}{2} \text{ or } \pm\tfrac{1}{2}$$

$$\therefore\ x=\tfrac{5}{2}\pm\tfrac{3}{2} \text{ or } \tfrac{5}{2}\pm\tfrac{1}{2}$$

$$=4 \text{ or } 1, \text{ or } 3 \text{ or } 2.$$

Ex. 11. Let $x^2-5x-6\sqrt{(x^2-5x+10)}+18=0$, to find x.

$$x^2-5x+10-6\sqrt{(x^2-5x+10)}+9=-8+9=1.$$

Extracting square root, $\sqrt{(x^2-5x+10)}-3=\pm 1$.

$$\therefore\ \sqrt{(x^2-5x+10)}=3\pm 1=4 \text{ or } 2.$$
$$\therefore\ x^2-5x+10=16 \text{ or } 4.$$
$$\therefore\ x^2-5x=\pm 6.$$
$$\therefore\ x^2-5x+\tfrac{25}{4}=\tfrac{25}{4}\pm 6=\tfrac{49}{4} \text{ or } \tfrac{1}{4}.$$
$$\therefore\ x-\tfrac{5}{2}=\pm\tfrac{7}{2} \text{ or } \pm\tfrac{1}{2}.$$
$$\therefore\ x=\tfrac{5}{2}\pm\tfrac{7}{2} \text{ or } \tfrac{5}{2}\pm\tfrac{1}{2}.$$
$$\therefore\ x=6 \text{ or } -1, \text{ or } 3 \text{ or } 2.$$

Ex. 12. $(x^2-13a^2)^2+2ax(x^2-13a^2)+a^2(x^2-13a^2)=$ to find the values of x.

$$x^4-26a^2x^2+169a^4+2ax^3-26a^3x+a^2x^2-49a^4=0.$$
$$\therefore\ x^4+2ax^3+a^2x^2-26a^2(x^2+ax)+120a^4=0.$$
$$(x^2+ax)^2-26a^2(x^2+ax)+(13a^2)^2=(13a^2)^2-120a^4=49a^4$$
$$\therefore\ x^2+ax-13a^2=\pm 7a^2;$$
$$x^2+ax=6a^2 \text{ or } 20a^2.$$
$$x^2+ax+\tfrac{1}{4}a^2=(6+\tfrac{1}{4})a^2 \text{ or } (20+\tfrac{1}{4})a^2$$
$$=\tfrac{25}{4}a^2 \text{ or } \tfrac{81}{4}a^2.$$
$$\therefore\ x+\tfrac{1}{2}a=\pm\tfrac{5}{2}a \text{ or } \pm\tfrac{9}{2}a.$$
$$\therefore\ x=-\tfrac{1}{2}a\pm\tfrac{5}{2}a \text{ or } -\tfrac{1}{2}a\pm\tfrac{9}{2}a.$$
$$=2a \text{ or } -3a, \text{ or } 4a \text{ or } -5a.$$

Ex. 13. Let $\sqrt{(x^2-7x+11)}+\dfrac{x^2}{2}=\tfrac{7}{2}x-4$, to find the values of x.

$$2\sqrt{(x^2-7x+11)}+x^2-7x+8=0$$
$$x^2-7x+11+2\sqrt{(x^2-7x+11)}+1=4.$$
$$\therefore\ \sqrt{(x^2-7x+11)}+1=\pm 2$$
$$\therefore\ \sqrt{(x^2-7x+11)}=-1\pm 2=1 \text{ or } -3,$$
$$\therefore\ x^2-7x+11=1 \text{ or } 9.$$
$$\therefore\ x^2-7x=-10 \text{ or } -2.$$
$$x^2-7x+(\tfrac{7}{2})^2=\tfrac{49}{4}-10 \text{ or } \tfrac{49}{4}-2$$
$$=\tfrac{9}{4} \text{ or } \tfrac{41}{4}.$$
$$\therefore\ x-\tfrac{7}{2}=\pm\tfrac{3}{2} \text{ or } \pm\tfrac{1}{2}\sqrt{(41)}.$$
$$\therefore\ x=\tfrac{7}{2}\pm\tfrac{3}{2} \text{ or } \tfrac{7}{2}\pm\tfrac{1}{2}\sqrt{(41)}.$$
$$\therefore\ x=5 \text{ or } 2, \text{ or } \tfrac{1}{2}\{7\pm\sqrt{(41)}\}.$$

Ex. 14. Let $\sqrt{\{12x^2+49+8\sqrt{(x^2+6x-7)}\}}=4x+3$, to find x.

$$12x^2+49+8\sqrt{(x^2+6x-7)}=16x^2+24x+9\,;$$
$$4x^2+24x-40-8\sqrt{(x^2+6x-7)}=0$$
$$\therefore\ x^2+6x-10-2\sqrt{(x^2+6x-7)}=0.$$
$$\therefore\ x^2+6x-7-2\sqrt{(x^2+6x-7)}+1=3+1=4.$$
$$\therefore\ \sqrt{(x^2+6x-7)}-1=\pm 2.$$
$$\therefore\ \sqrt{(x^2+6x-7)}=1\pm 2=3 \text{ or } -1$$
$$\therefore\ x^2+6x-7=9 \text{ or } 1\,;$$
$$\therefore\ x^2+6x=16 \text{ or } 8.$$
$$\therefore\ x^2+6x+9=25 \text{ or } 17.$$
$$x+3=\pm 5 \text{ or } \pm\sqrt{(17)}.$$
$$\therefore x=-3\pm 5 \text{ or } -3\pm\sqrt{(17)}.$$
$$\therefore\ x=2 \text{ or } -8, \text{ or } -3\pm\sqrt{(17)}.$$

Ex. 15. Let $2x^4-x^2+8-9x^3+18x=0$, to find the values of x.

Divide by $2x^2$, $x^2-\frac{1}{2}+\frac{4}{x^2}-\frac{9}{2}x+\frac{9}{x}=0$;

$$x^2-4+\frac{4}{x^2}-\frac{9}{2}\left(x-\frac{2}{x}\right)+\frac{7}{2}=0.$$
$$\therefore\ \left(x-\frac{2}{x}\right)^2-\frac{9}{2}\left(x-\frac{2}{x}\right)+\frac{81}{16}=\frac{81}{16}-\frac{7}{2}=\frac{25}{16}$$
$$\therefore\ x-\frac{2}{x}-\frac{9}{4}=\pm\frac{5}{4}.$$
$$x-\frac{2}{x}=\frac{9}{4}\pm\frac{5}{4}=\frac{7}{2}\text{or } 1\,;$$
$$x^2-2=\frac{7}{2}x \text{ or } x$$
$$\therefore x^2-\frac{7}{2}x=2 \text{ or } x^2-x=2.$$

$$\begin{array}{ll}
x^2-\frac{7}{2}x+\frac{49}{16}=\frac{49}{16}+2 & \qquad x^2-x+\frac{1}{4}=2+\frac{1}{4} \\
x^2-\frac{7}{2}x+\frac{49}{16}=\frac{81}{16} & \qquad x^2-x+\frac{1}{4}=\frac{9}{4} \\
x-\frac{7}{4}=\pm\frac{9}{4} & \qquad x-\frac{1}{2}=\pm\frac{3}{2} \\
x=\frac{7}{4}\pm\frac{9}{4}=4 \text{ or } -\frac{1}{2} & \qquad x=\frac{1}{2}\pm\frac{3}{2}=2 \text{ or } -1.
\end{array}$$

Ex. 16. Let $x^2-4x\sqrt{x}+7x-24\sqrt{x}+36=0$, to find the values of x.

Divide by x, $x-4\sqrt{x}+7-\dfrac{24}{\sqrt{x}}+\dfrac{36}{x}=0.$

$$x+12+\frac{36}{x}-4\left(\sqrt{x}+\frac{6}{\sqrt{x}}\right)=5.$$

$$\left(\sqrt{x}+\frac{6}{\sqrt{x}}\right)^2-4\left(\sqrt{x}+\frac{6}{\sqrt{x}}\right)+4=9.$$

$$\sqrt{x}+\frac{6}{\sqrt{x}}-2=\pm 3.$$

$$\sqrt{x}+\frac{6}{\sqrt{x}}=2\pm 3=5 \text{ or } -1$$

$$x-5\sqrt{x}=-6 \text{ or } x+\sqrt{x}=-6.$$

$$x-5\sqrt{x}+\tfrac{25}{4}=\tfrac{25}{4}-6=\tfrac{1}{4} \qquad x+\sqrt{x}+\tfrac{1}{4}=-6+\tfrac{1}{4}=-\tfrac{23}{4}.$$

$$\sqrt{x}-\tfrac{5}{2}=\pm\tfrac{1}{2} \qquad \sqrt{x}+\tfrac{1}{2}=\pm\sqrt{(-\tfrac{23}{4})}.$$

$$\sqrt{x}=\tfrac{5}{2}\pm\tfrac{1}{2}=3 \text{ or } 2 \qquad \therefore \sqrt{x}=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{(-23)}.$$

$$x=9 \text{ or } 4 \qquad \therefore x=\{-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{(-23)}\}^2.$$

190. If all the terms of an equation be brought to one side and if the expression thus obtained can be represented as the product of simple or quadratic factors, then the solution of the equation can be effected by the methods already given.

Ex. Let $(x-a)(x^2-3bx+2b^2)=0.$

Dividing both sides of the equation by each factor we may see that other factor is zero ; $\therefore$ $x-a=0$ or $x^2-3bx+2b^2=0$. From the first equation we get $x=a$ and from the second we get $x=b$ or $2b$.

Ex. 17. Let $x^3=7x+6$, to find the values of x.

$$x(x^2-1)-6(x+1)=0.$$

$$\therefore (x+1)(x^2-x-6)=0.$$

$$\therefore x+1=0 \text{ and } x^2-x-6=0.$$

$$\therefore x=-1 \text{ and } x^2-x+\tfrac{1}{4}=6\tfrac{1}{4}=\tfrac{25}{4}.$$

$$\therefore x-\tfrac{1}{2}=\pm\tfrac{5}{2}.$$

$$\therefore x=\tfrac{1}{2}\pm\tfrac{5}{2}=3 \text{ or } -2.$$

$$\therefore x=-1, 3 \text{ or } -2.$$

Ex. 18. To solve $\sqrt{(a^2-4ax+3x^2)}+\sqrt{(a^2-5ax+6x^2)}$
$=\sqrt{(2a^2-9ax+9x^2)}.$

$$\sqrt{\{(a-3x)(a-x)\}}+\sqrt{\{(a-3x)(a-2x)\}}$$
$$-\sqrt{\{(a-3x)(2a-3x)\}}=0.$$

$$\sqrt{(a-3x)}\{\sqrt{(a-x)}+\sqrt{(a-2x)}-\sqrt{(2a-3x)}\}=0.$$

$\therefore \sqrt{(a-3x)}=0,\ a-3x=0;\ \therefore x=\frac{1}{3}a.$

Also $\sqrt{(a-x)}+\sqrt{(a-2x)}=\sqrt{(2a-3x)}$.

Squaring $a-x+a-2x+2\sqrt{\{(a-x)(a-2x)\}}=2a-3x$.

$\therefore 2\sqrt{\{(a-x)(a-2x)\}}=0;\ \therefore \sqrt{(a-x)}=0$ or $\sqrt{(a-2x)}=0$.

$\therefore x=a$ or $x=\frac{1}{2}a$.

$\therefore x=a,\ \frac{1}{2}a$ or $\frac{1}{3}a$.

Ex. 19. Let $x^3-3x-2=0$, to find the values of x.

$x(x^2-1)-2(x-1)=0.$

$\therefore (x-1)\{x^2+x-2\}=0.$

$\therefore x-1=0,\ x=1.$

Also $x^2+x-2=0;\ \therefore x^2+x+\frac{1}{4}=2+\frac{1}{4};$

$\therefore x+\frac{1}{2}=\pm\frac{3}{2};\ \therefore x=-\frac{1}{2}\pm\frac{3}{2}=1$ or -2.

Ex. 20. Let $a=\sqrt[3]{1}$, to find the values of a or to find the cube root of unity.

$a^3=1.$

$\therefore a^3-1=0$, or $(a-1)(a^2+a+1)=0$.

$\therefore a-1=0$; and $a=1$.

Also $a^2+a+1=0$;

hence $a=\frac{1}{2}\{-1\pm\sqrt{(-3)}\}$

Ex. 21. $\frac{x}{b}+\frac{a}{x}+\frac{a^2}{bx}+\frac{a^3}{b^2x}=1+\frac{a}{b}+\frac{a^2}{b^2}+\frac{a^3}{bx^2}$, to find x.

$$\frac{x-b}{b}+\frac{a(b-x)}{bx}+\frac{a^2(b-x)}{b^2x}+\frac{a^3(x-b)}{b^2x^2}=0.$$

$$\therefore (x-b)\left\{\frac{1}{b}-\frac{a}{bx}-\frac{a^2}{b^2x}+\frac{a^3}{b^2x^2}\right\}=0.$$

$\therefore x-b=0$ and $x=b$.

Also $\frac{1}{b}-\frac{a}{bx}-\frac{a^2}{b^2x}+\frac{a^3}{b^2x^2}=0.$

$\therefore bx^2-abx-a^2x+a^3=0.$

$\therefore bx(x-a)-a^2(x-a)=0.$

$\therefore (bx-a^2)(x-a)=0.$

$\therefore x-a=0$ or $bx-a^2=0$.

$\therefore x=a$, or $x=\frac{a^2}{b}$.

Ex. 22. Let $(x+1)^4=8(x^4+1)$, to find the values of x.

$$x^4+4x^3+6x^2+4x+1=8x^4+8.$$

$$\therefore\ 7x^4-4x^3-6x^2-4x+7=0.$$

This may be solved by Art. 189 or by Art. 190. We will solve it according to Art. 190.

$$7(x^4+1)-4(x^3+x)-6x^2=0.$$

Divide by $7x^2$, then $x^2+\dfrac{1}{x^2}-\dfrac{4}{7}\left(x+\dfrac{1}{x}\right)-\dfrac{6}{7}=0.$

$$\therefore\ x^2+2+\frac{1}{x^2}-\frac{4}{7}\left(x+\frac{1}{x}\right)+(\tfrac{2}{7})^2=2\tfrac{6}{7}+(\tfrac{2}{7})^2.$$

$$(x+\tfrac{1}{x})^2-\tfrac{4}{7}(x+\tfrac{1}{x})+(\tfrac{2}{7})^2=\tfrac{144}{49}.$$

$$\therefore\ x+\tfrac{1}{x}-\tfrac{2}{7}=\pm\tfrac{12}{7}.$$

$$\therefore\ x+\tfrac{1}{x}=\tfrac{2}{7}\pm\tfrac{12}{7}=2 \text{ or } -\tfrac{10}{7}.$$

$$\therefore\ x^2-2x+1=0, \text{ or } x^2+\tfrac{10}{7}x+1=0.$$

From the first equation $x=1$; from the second

$$x=-\tfrac{1}{7}\{5\pm\sqrt{(-24)}\}.$$

Ex. 23. Let $x^5-a^5=0$, to find the values of x.

$$x^5-a^5=(x-a)(x^4+ax^3+a^2x^2+a^3x+a^4)=0.$$

$$\therefore\ x-a=0 \text{ and } x=a.$$

Also $x^4+ax^3+a^2x^2+a^3x+a^4=0$;

divide by x^2, $x^2+ax+a^2+\dfrac{a^3}{x}+\dfrac{a^4}{x^2}=0.$

$$\therefore\ x^2+\frac{a^4}{x^2}+a\left(x+\frac{a^2}{x}\right)+a^2=0.$$

$$\therefore\ x^2+2a^2+\frac{a^4}{x^2}+a\left(x+\frac{a^2}{x}\right)+\frac{a^2}{4}=a^2+\frac{a^2}{4}=\frac{5a^2}{4}.$$

$$\therefore\ x+\frac{a^2}{x}+\frac{a}{2}=\pm\frac{a\sqrt{5}}{2}.$$

$$\therefore\ x+\frac{a^2}{x}=-\frac{a}{2}\pm\frac{a\sqrt{5}}{2},$$

$$\therefore\ x^2+\tfrac{1}{2}a(1-\sqrt{5})x+a^2=0, \text{ or } x^2+\tfrac{1}{2}a(1+\sqrt{5})x+a^2=0; \text{ etc.}$$

191. We will now solve some miscellaneous examples reducible to quadratics.

Ex. 24. Let $\dfrac{x+\sqrt{(x^2-a^2)}}{x-\sqrt{(x^2-a^2)}}=\dfrac{x}{a}$, to find the values of x.

$$\frac{x}{\sqrt{(x^2-a^2)}}=\frac{x+a}{x-a}; \quad \therefore\ x^2(x-a)^2=(x^2-a^2)(x+a)^2.$$

$$\therefore\ (x-a)\{x^2(x-a)-(x+a)^3\}=0.$$

$$\therefore\ x-a=0 \text{ and } x=a;$$

Also $x^2(x-a)-(x+a)^3=0$;

$$\therefore\ x^3-ax^2-x^3-3ax^2-3a^2x-a^3=0;$$

$$\therefore\ 4ax^2+3a^2x+a^3=0;$$

$$x^2+\tfrac{3}{4}ax+\tfrac{9}{64}a^2=-\frac{a^2}{4}+\tfrac{9}{64}a^2=-\tfrac{7}{64}a^2.$$

$$\therefore\ x+\tfrac{3}{8}a=\pm\tfrac{1}{8}a\sqrt{(-7)};$$
$$\therefore\ x=\tfrac{1}{8}a\{-3\pm\sqrt{(-7)}\}$$

Ex. 25. Let $\tfrac{3}{4}x^2=\dfrac{5x-8}{x^2-5x+8}-\dfrac{5x-2}{x^2-5x+2}$,

to find the values of x.

$$\tfrac{3}{4}x^2=\frac{5x-8}{x^2-5x+8}+1-\left(\frac{5x-2}{x^2-5x+2}+1\right)$$

$$\therefore\ \tfrac{3}{4}x^2=\frac{x^2}{x^2-5x+8}-\frac{x^2}{x^2-5x+2}; \quad \therefore\ x=0;$$

$$\text{and}\ \tfrac{3}{4}=\frac{1}{x^2-5x+8}-\frac{1}{x^2-5x+2}.$$

$$\tfrac{3}{4}=\frac{-6}{(x^2-5x+8)(x^2-5x+2)}.$$

$$\therefore\ (x^2-5x+8)(x^2-5x+2)=-8.$$

$$\therefore\ \{(x^2-5x+5)+3\}\{(x^2-5x+5)-3\}=-8.$$

$$\therefore\ (x^2-5x+5)^2-9=-8.$$

$$\therefore\ (x^2-5x+5)^2=1.$$

$$\therefore\ x^2-5x+5=\pm 1.$$

$$\therefore\ x^2-5x+4=0,\ x^2-5x+6=0.$$

$$\therefore\ x=1,\ 4,\ 3 \text{ or } 2.$$

Ex. 26. Let $\dfrac{\sqrt{(1+x)}}{1+\sqrt{(1+x)}}=\dfrac{\sqrt{(1-x)}}{1-\sqrt{(1-x)}}$,
to find the values of x.

$$\frac{\sqrt{(1+x)}}{\sqrt{(1+x)}+1}\times\frac{\sqrt{(1+x)}-1}{\sqrt{(1+x)}-1}=\frac{\sqrt{(1-x)}}{1-\sqrt{(1-x)}}\times\frac{1+\sqrt{(1-x)}}{1+\sqrt{(1-x)}}$$

$$\therefore \frac{1+x-\sqrt{(1+x)}}{x}=\frac{\sqrt{(1-x)}+1-x}{x}$$

$$\therefore 2x=\sqrt{(1+x)}+\sqrt{(1-x)}.$$

$$\therefore 4x^2=1+x+1-x+2\sqrt{(1-x^2)}.$$

$$4x^2-2=2\sqrt{(1-x^2)}.$$

$$\therefore 2x^2-1=\sqrt{(1-x^2)}.$$

$$\therefore 4x^4-4x^2+1=1-x^2.$$

$$\therefore 4x^4=3x^2;\ \therefore x^2=0;\ \therefore x=0.$$

Also $4x^2=3$; $\therefore x=\pm\frac{1}{2}\sqrt{(3)}$.

Ex. 27. Let $\sqrt{x}-\dfrac{8}{x}=\dfrac{7}{\sqrt{x}-2}$, to find the values of x.

Multiply by $\sqrt{x}-2$,

$$x-2\sqrt{x}-\frac{8}{\sqrt{x}}+\frac{16}{x}=7.$$

$$x+8+\frac{16}{x}-2\left(\sqrt{x}+\frac{4}{\sqrt{x}}\right)+1=16.$$

$$\therefore \left(\sqrt{x}+\frac{4}{\sqrt{x}}\right)^2-2\left(\sqrt{x}+\frac{4}{\sqrt{x}}\right)+1=16.$$

$$\therefore \sqrt{x}+\frac{4}{\sqrt{x}}-1=\pm 4.$$

$\therefore x-5\sqrt{x}+4=0$ or $x+3\sqrt{x}+4=0$

$x-5\sqrt{x}+(\frac{5}{2})^2=\frac{25}{4}-4=\frac{9}{4}$	$x+3\sqrt{x}+(\frac{3}{2})^2=\frac{9}{4}-4=-\frac{7}{4}.$
$\sqrt{x}-\frac{5}{2}=\pm\frac{3}{2}$	$\sqrt{x}+\frac{3}{2}=\pm\frac{1}{2}\sqrt{(-7)}.$
$\sqrt{x}=\frac{5}{2}\pm\frac{3}{2}$	$\sqrt{x}=-\frac{3}{2}\pm\frac{1}{2}\sqrt{(-7)}$
$\therefore \sqrt{x}=4$ or 1	$\therefore x=\frac{1}{4}\{-3\pm\sqrt{(-7)}\}^2$
$\therefore x=16$ or 1	$=\frac{1}{2}\{1\pm3\sqrt{(-7)}\}$

Ex. 28. To solve.

$$\frac{x-\sqrt{(x^2-2)}}{x+\sqrt{(x^2-2)}}=2x\sqrt[4]{(2x)}\times(x^2-2)^{\frac{5}{8}}\left\{\frac{\sqrt{(x+\sqrt{2})}+\sqrt{(x-\sqrt{2})}}{\sqrt{(x+\sqrt{2})}-\sqrt{(x-\sqrt{2})}}\right\}^{\frac{1}{2}}$$

$$\left\{\frac{\sqrt{(x+\sqrt{2})}-\sqrt{(x-\sqrt{2})}}{\sqrt{(x+\sqrt{2})}+\sqrt{(x-\sqrt{2})}}\right\}^{2}$$

$$=2x\sqrt[4]{(2x)}\times(x^2-2)^{\frac{5}{8}}\left\{\frac{\sqrt{(x+\sqrt{2})}+\sqrt{(x-\sqrt{2})}}{\sqrt{(x+\sqrt{2})}-\sqrt{(x-\sqrt{2})}}\right\}^{\frac{1}{2}}.$$

$$\left\{\frac{\sqrt{(x+\sqrt{2})}-\sqrt{(x-\sqrt{2})}}{\sqrt{(x+\sqrt{2})}+\sqrt{(x-\sqrt{2})}}\right\}^{\frac{5}{2}}=(2x)^{\frac{5}{4}}\times(x^2-2)^{\frac{5}{8}}.$$

Extract the fifth root of both the sides and raise them to the fourth power,

$$\therefore\ \frac{x-\sqrt{(x^2-2)}}{x+\sqrt{(x^2-2)}}=2x\sqrt{(x^2-2)}.$$

$$\frac{x-\sqrt{(x^2-2)}}{x+\sqrt{(x^2-2)}}\times\frac{x-\sqrt{(x^2-2)}}{x-\sqrt{(x^2-2)}}=2x\sqrt{(x^2-2)}.$$

$$\therefore\ \frac{\{x-\sqrt{(x^2-2)}\}^2}{2}=2x\sqrt{(x^2-2)}.$$

$$x^2-1-x\sqrt{(x^2-2)}=2x\sqrt{(x^2-2)}.$$

$$x^2-1=3x\sqrt{(x^2-2)}.$$

$$x^4-2x^2+1=9x^4-18x^2$$

$$8x^4-16x^2-1=0.$$

$$x^4-2x^2+1=\tfrac{1}{8}+1=\tfrac{9}{8}.$$

$$x^2-1=\pm\tfrac{3}{2}\sqrt{(\tfrac{1}{2})}.$$

$$\therefore\ x^2=1\pm\tfrac{3}{2}\sqrt{(\tfrac{1}{2})}.$$

$$\therefore\ x=\pm\sqrt{\{1\pm\tfrac{3}{2}\sqrt{(\tfrac{1}{2})}\}}.$$

Ex. 29. $\dfrac{1}{15x^2-11x+2}+\dfrac{1}{10x^2-9x+2}=18x^2-9x+1$, to find the values of x.

$$\frac{1}{(5x-2)(3x-1)}+\frac{1}{(5x-2)(2x-1)}=(3x-1)(6x-1).$$

$$\frac{5x-2}{(5x-2)(3x-1)(2x-1)}=(3x-1)(6x-1).$$

$$(3x-1)^2(2x-1)(6x-1)=1.$$
$$(3x-1)^2(6x-3)(6x-1)=3.$$
$$(3x-1)^2\{2(3x-1)-1\}\{2(3x-1)+1\}=3$$
$$(3x-1)^2\{4(3x-1)^2-1\}=3.$$
$$4(3x-1)^4-(3x-1)^2=3$$
$$(3x-1)^4-\tfrac{1}{4}(3x-1)^2+\tfrac{1}{64}=\tfrac{3}{4}+\tfrac{1}{64}=\tfrac{49}{64}.$$
$$(3x-1)^2-\tfrac{1}{8}=\pm\tfrac{7}{8}.$$
$$(3x-1)^2=\tfrac{1}{8}\pm\tfrac{7}{8}=1 \text{ or } -\tfrac{3}{4}$$
$$3x-1=\pm 1 \text{ or } \pm\tfrac{1}{2}\sqrt{(-3)}.$$
$$\therefore\ 3x=1\pm 1 \text{ or } 1\pm\tfrac{1}{2}\sqrt{(-3)}.$$
$$\therefore\ x=0 \text{ or } x=\tfrac{2}{3}, \text{ or } x=\tfrac{1}{3}\{1\pm\tfrac{1}{2}\sqrt{(-3)}\}.$$

EXAMPLES LXVII.

1. $x^4-5x^2+4=0.$
2. $x^6-9x^3+8=0.$
3. $x^{10}-33x^5+32=0.$
4. $x-3\sqrt{x}+2=0.$
5. $x-14\sqrt{x}+45=0.$
6. $2x^3+3\sqrt{(x^3)}=152.$
7. $x^{2m}-5x^m+4=0.$
8. $x^4-41x^2+400=0.$
9. $x+3\sqrt{(ax)}=10a.$
10. $2x^4-3x^2=1175.$
11. $x^{\frac{2}{3}}+7x^{\frac{1}{3}}=44.$
12. $x^{\frac{2}{m}}-8x^{\frac{1}{m}}+7=0.$
13. $\sqrt[3]{x}+\dfrac{2}{3\sqrt[3]{x}}=2\tfrac{1}{3}.$
14. $2\sqrt[5]{(x^2)}=\dfrac{5}{\sqrt[5]{(x^{-1})}}-2.$
15. $33x^{-2}-14x^{-1}+1=0.$
16. $x^{-2}+\tfrac{5}{6}x^{-1}=\tfrac{2}{3}.$
17. $x^n\sqrt{x^n}=\dfrac{9x^n}{\sqrt[4]{x^n}}-8.$
18. $x^{-2n}-x^{-n}=2.$
19. $2(x^{\frac{1}{4}}-x^{-\frac{1}{4}})=3.$
20. $8(x^{\frac{3n}{4}}+x^{-\frac{3n}{4}})=65.$
21. $\sqrt{(x^2-9)}-3\sqrt[4]{(x^2-9)}+2=0.$
22. $\sqrt{\{8(x^2-5)\}}-4\sqrt[4]{\{8(x^2-5)\}}+3=0.$
23. $3\sqrt[4]{(x^2+7)}+\sqrt{(x^2+7)}=10.$
24. $x^2+x+2\sqrt{(x^2+x+4)}=20.$
25. $2x^2-5x+12-5\sqrt{(2x^2-5x+6)}=0.$

26. $x^2+\frac{4}{x^2}-\frac{15x}{2}-\frac{15}{x}+\frac{35}{2}=0.$

27. $(x-1)(x-2)(x-3)(x-4)=120.$

28. $(x-2a)(x-3a)(x-4a)(x-5a)=360a^4.$

29. $(x^2-5ax+2a^2)^2+6(x^2-5ax+2a^2)a^2+8a^4=0.$

30. $6x^4+62a^2x^2+6a^4-35(x^2+a^2)ax=0.$

31. $8x^2-54x\sqrt{x}+101x-54\sqrt{x}+8=0.$

32. $x^2-\frac{1}{3}a\sqrt{(3x^2-4ax+4a^2)}=\frac{2}{3}(2x+13a)a.$

33. $x^2-2\sqrt{(3x^2-2ax+4)}+4=\frac{2}{3}a(x+\frac{1}{2}a+1).$

34. $\sqrt{(x^4-1)}+\sqrt{(x^2-1)}=x^3.$ 35. $3^{x+1}+9^x=108.$

36. $x-4+\sqrt{\left(\frac{x-4}{x+4}\right)}=\frac{12}{x+4}.$

37. $(x^{p+r})^{\frac{1}{2pr}}=\frac{2}{5}\left(\frac{a^2-b^2}{a^2+b^2}\right)(x^{\frac{1}{p}}+x^{\frac{1}{r}}).$

38. $2\sqrt{x}-\sqrt{\{4x-\sqrt{(11x+5)}\}}=1.$

39. $x^2(2x-1)^2+8x(x-1)^2-3(2x+1)^2+4(2x+1)+3=0.$

40. $x-2\sqrt{x}+\frac{16}{\sqrt{x}}+\frac{60}{x}=15.$ 41. $\frac{12}{x^2-x-2}=9+x-x^2.$

42. $x^2-x+5\sqrt{(2x^2-5x+6)}=\frac{1}{2}(3x+33).$

43. $\frac{x}{x^2+x+5}+\frac{5}{\sqrt{(x^2+x+5)}}=\frac{116}{25x}.$

44. $\frac{x^2+1}{x}+\frac{7x+7}{2\sqrt{x}}=13.$ 45. $\frac{2}{x^2+2x-2}+\frac{3}{x^2-2x+3}=\frac{x}{2}.$

46. $x^3-3x^2+2x=0.$ 47. $x^2-2x=4.$

48. $\sqrt[3]{(x^{\frac{1}{2}}+22)}-\sqrt[3]{(x^{\frac{1}{2}}-4)}=2.$

49. $1+\sqrt[3]{\{(2x+7)(4\sqrt{x}-7)\}}=2\sqrt{x}.$

50. $(x-1)(x-3)(x-5)(x-7)=(x-2)(x-4)(x-6)(x-5).$

51. $\sqrt[4]{(x+11)}+\sqrt{(x+11)}=6.$

52. $x^2+12x-365=2\sqrt{(x^2+12x-5)}.$

53. $\sqrt{(a^2+2ax-3x^2)}-\sqrt{(a^2+ax-6x^2)}$
$$=\sqrt{\{(a+3x)(2a-3x)\}}.$$

54. $7x+\sqrt{(x^2-17x+4)}=2x^2-27x+2.$

55. $x^4-5x^3+5x^2-5x+4=0.$

56. $x^4+2x^3-11x^2+4x+4=0.$

57. $6\sqrt{(x^2-2x+6)}=21+2x-x^2.$

58. $\sqrt[4]{(\frac{41}{16}+x)}+\sqrt{(\frac{41}{16}+x)}=6.$

59. $\sqrt[4]{\left(\frac{41}{16}+x\right)}-\sqrt[4]{\left(\frac{41}{16}-x\right)}=1.$

60. $\dfrac{1}{4x^2-9x+5}+\dfrac{1}{4x^2-7x+3}=8(x-1)^2.$

61. $\dfrac{4}{x^2-4x+3}-\dfrac{2}{x^2-3x+2}=x^2-3x+2.$

62. $\dfrac{1}{6x^2-7x+2}+\dfrac{1}{12x^2-17x+6}=8x^2-6x+1.$

63. $\dfrac{5}{x^2-5x+4}+\dfrac{5}{x^2-11x+28}=x^2-8x+10.$

64. $\sqrt{(x^2-4x+12)}-\frac{1}{2}(x^2+1)=4-2x.$

65. $\sqrt{(x^2-4x+16)}-\sqrt{(x^2-4x+14)}=\sqrt{(21)}-\sqrt{(19)}.$

66. $\sqrt{\{a^2+(x+b)a+bx\}}+\sqrt{\{b^2-(x-a)b-ax\}}$
$$=\sqrt{\{ab-(a-b)x-x^2\}}.$$

67. $x^4-2x^3+x-132=0.$

68. $x(x+4)+\frac{1}{x}(\frac{1}{x}+4)=14\frac{1}{4}.$

69. $\sqrt{(5+\sqrt{x})}+\sqrt{(5-\sqrt{x})}=\dfrac{6}{\sqrt{(5+\sqrt{x})}}.$

70. $\dfrac{x+6}{x-8}-\dfrac{x-8}{x+6}=\dfrac{7(x-1)}{x^2-2x-72}.$ 71. $x+\dfrac{2}{\sqrt{x}}=3.$

72. $(x+4)(x-3)=36-\sqrt{\{(x+3)(x-2)\}}.$

73. $\frac{1}{2}\sqrt{(x^2+\frac{3}{2}x+\frac{7}{2})}+\frac{1}{3}(2x^2+3x)=16.$

74. $(x-4)^2+2(x-4)=\dfrac{4}{x}-1.$

75. $\sqrt{(x^2-6x+15)}-\sqrt{(x^2-6x+13)}=\sqrt{(10)}-\sqrt{(8)}.$

76. $\frac{1}{4}(10x^3+x+5)^2+(x^2+7x-2)^2=(3x^2-x+4)^2$
$+\frac{1}{4}(8x^2+10x-4)^2.$

77. $16x(x-1)(x-2)(x-3)=9.$

78. $(12x-5)(2x-1)(6x-1)(4x-3)=35.$

79. $(x-a)(x-2a)(x-3a)(x-4a)=3a^4.$

80. $(12x-1)(6x-1)(4x-1)(3x-1)=5.$

81. $\sqrt{(x+2\sqrt{x})}-\sqrt{(x-2\sqrt{x})}=2\sqrt[3]{(x^2-4x)}.$

82. $\sqrt{(1-x)}+\sqrt{\{1-x+\sqrt{(1+x)}\}}=\sqrt{(1+x)}.$

83. $\dfrac{40}{x^2-49}-\dfrac{20}{x^2+7x}+\dfrac{8}{x^2+8x-9}-1=\dfrac{12}{x^2+3x-54}.$

84. $(x-1)^2+(a-1)^2=2(ax+1)+\sqrt{\{3(x+a)^2+4ax\}}.$

85. $\sqrt{(x+\sqrt{x})}-2\sqrt{(x-\sqrt{x})}=a\left(\dfrac{x}{x+\sqrt{x}}\right)^{\frac{1}{2}}.$

86. $\sqrt{\left(\dfrac{x^2-2x+3}{x^2+2x+4}\right)}+\sqrt{\left(\dfrac{x^2+2x+4}{x^2-2x+3}\right)}=2\tfrac{1}{2}.$

87. $\dfrac{8}{(x-3)(x+1)}-\dfrac{6}{(x-2)(x+1)}-\dfrac{1}{(x-1)(x-2)}$
$-\dfrac{2}{(x-2)(x-4)}+\dfrac{5}{12}=0.$

88. $\dfrac{3}{x-1}+\dfrac{6}{x+2}+\dfrac{1}{x-3}=\dfrac{5}{x+1}+\dfrac{2}{x-2}+\dfrac{7}{x+3}.$

89. $\dfrac{5}{x-1}+\dfrac{4}{x+2}+\dfrac{21}{x-3}=\dfrac{5}{x+1}+\dfrac{4}{x-2}+\dfrac{21}{x+3}.$

90. $x^2-5x=\sqrt{(50x-24-10x^2)}.$

91. $x^2-2x=\sqrt{(8x^2-16x+9)}.$

92. $(a+b)\sqrt{(a^2+b^2+x^2)}-(a-b)\sqrt{(a^2+b^2-x^2)}=a^2+b^2.$

93. $\dfrac{\sqrt{(1+x)}}{1+\sqrt{(1+x)}}=\dfrac{\sqrt{(1-x)}}{1-\sqrt{(1-x)}}$.

94. $\dfrac{\sqrt{(x^2+3a)}+\sqrt{(x^2-3a)}}{\sqrt{(x^2+3a)}-\sqrt{(x^2-3a)}}=\dfrac{3x+3\sqrt{a}}{3x-3\sqrt{a}}$.

95. $\sqrt{(x^2-5ax+4a^2)}-2\sqrt{(x^2-5ax-4a^2)}$
$$=\sqrt{(3x^2-15ax-14a^2)}.$$

96. $\sqrt{(x^2-3ax+2a^2)}+\sqrt{(x^2+3ax+2a^2)}=\sqrt{(2x^2+4a^2)}$.

97. $\dfrac{\sqrt{(x^2+a^2)}+\sqrt{(x^2-a^2)}}{\sqrt{(x^2+a^2)}-\sqrt{(x^2-a^2)}}+\dfrac{\sqrt{(x^2+a^2)}-\sqrt{(x^2-a^2)}}{\sqrt{(x^2+a^2)}+\sqrt{(x^2-a^2)}}$
$$=\frac{25}{6a}\sqrt{(x^2-a^2)}.$$

98. $\dfrac{\sqrt{(x+1)}+\sqrt{(x-1)}}{\sqrt{(x+1)}-\sqrt{(x-1)}}+\dfrac{\sqrt{(x+1)}-\sqrt{(x-1)}}{\sqrt{(x+1)}+\sqrt{(x-1)}}=2(\sqrt{x}+\frac{3}{2})$.

√99. $2x-1+x\sqrt{(x^2-3)}+(x-1)\sqrt{(x^2-2x-2)}=0$.

/100. $\dfrac{x^3-3x+(x^2-1)\sqrt{(x^2-4)}-2}{x^3-3x+(x^2-1)\sqrt{(x^2-4)}+2}=\sqrt{(\frac{4}{5})}$;

101. $\sqrt{x}+2\sqrt{(2a-x)}=\sqrt{\{x+\sqrt{(2a^2-3ax+x^2)}\}}$.

102. $\dfrac{(b-c)^2}{(b-c)^2-(x-a)^2}+\dfrac{(c-a)^2}{(c-a)^2-(x-b)^2}$
$$+\frac{(a-b)^2}{(a-b)^2-(x-c)^2}=2.$$

103. $\sqrt{\{x(m+n-x)\}}+\sqrt{\{m(n+x-m)\}}+\sqrt{\{n(m+x-n)\}}=0$.

104. $(x^2-2x+2)^2=4(2x-3)$.

105. $\left(\dfrac{x^2-13x+31}{x^2-x-11}\right)^2=1-\dfrac{4x-8}{x+1}$.

106. $(x^2-7x-8)^2=(x+1)(x+4)^3$.

107. $(4x^2-7ax-2a^2)^2=4(4x-a)(x-2a)^3$.

108. $\frac{1}{6}x = \dfrac{x-1}{x^2-6x+6} + \dfrac{x^2-5x+6}{6(x^2-6x+7)}$.

109. $3x-12 = \dfrac{x^2-30x+12}{2x^2+2x-1} - \dfrac{x^2-3x}{x^2-3x+1}$.

110. $\dfrac{2a^2b^2(x^4+1)}{2a^2+b^2} + \left(2a^2+b^2+\dfrac{5a^2b^2}{2a^2+b^2}\right)x^2 = 3abx(x^2+1)$.

111. $\dfrac{(x+1)(x+2)}{(x-1)(x-2)} + \dfrac{(x-1)(x-2)}{(x+1)(x+2)}$

$$= \frac{(x+3)(x+4)}{(x-3)(x-4)} + \frac{(x-3)(x-4)}{(x+3)(x+4)}.$$

112. $(x+1)^5+(x-1)^5 = 10\{(x+1)^3+(x-1)^3\}$.

113. $(x-1)(x-2)(x-3)(x-4)$

$$=1^2+(x-1)^2+(x-2)^2+(x-3)^2+(x-4)^2+5^2.$$

114. $2b\{\sqrt{(1+x)}-1\}+2c\{\sqrt{(1-x)}+1\}$

$$=2b\sqrt{(1-x)}+2c\sqrt{(1+x)}+\sqrt{(1-x^2)}.$$

115. $\left(\dfrac{x+2}{x+3}\right)^2 + \left(\dfrac{x-2}{x-3}\right)^2 = \frac{13}{6} \times \dfrac{x^2-4}{x^2-9}$.

116. $\dfrac{x-\sqrt{(x^2-a^2)}}{\sqrt{\{x+\sqrt{(x^2-a^2)}\}}} = \sqrt[4]{(x^2-a^2)}\{\sqrt{(x^2+ax)}-\sqrt{(x^2-ax)}\}$

117. $(1-x)x\sqrt{(1+x)}-(1+x)x\sqrt{(1-x)}$

$$=\tfrac{3}{4}[\sqrt{\{2+2\sqrt{(1-x^2)}\}}].$$

118. $\sqrt[8]{(x^2-1)}\sqrt[8]{(x^2+1)^5}-\sqrt[8]{(x^2+1)}\sqrt[8]{(x^2-1)^5}=\sqrt{(2x)}$.

119. $(a^2-ax+x^2)^2=\frac{7}{3}(a^4+a^2x^2+x^4)$.

120. $\sqrt{(1-a)}\times\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{4}}+\sqrt{(1+a)}\times\left(\dfrac{1-x}{1+x}\right)^{\frac{1}{4}}=2\sqrt[4]{(1-a^2)}$.

Simultaneous Equations Involving Quadratics.

191 When there are more equations and unknown quantities than one, a single equation, involving only one of the unknown quantities, may sometimes be obtained by the rules laid down for the solution of simple equations, and one of the unknown quantities being discovered, the others may be obtained by substituting its value in the preceding equations.

Ex. 1. $\left.\begin{aligned} x-\frac{x-y}{2}&=4. \\ y-\frac{x+3y}{x+2}&=1. \end{aligned}\right\}$ to find x and y.

From the first equation, $2x-x+y=8$.

$$\text{or } x+y=8.$$
$$\therefore x=8-y.$$

From the second equation, $xy+2y-x-3y=x+2$.

$$\text{or } xy-2x-y=2.$$

By substitution, $(8-y)\times y-2(8-y)-y=2$.

$$8y-y^2-16+2y-y=2.$$
$$\therefore y^2-9y=-18$$
$$\therefore y^2-9y+(\tfrac{9}{2})^2=(\tfrac{9}{2})^2-18=\tfrac{9}{4};$$
$$y-\tfrac{9}{2}=\pm\tfrac{3}{2}.$$
$$\therefore y=\tfrac{9}{2}\pm\tfrac{3}{2}=6 \text{ or } 3.$$
$$x=8-y=2 \text{ or } 5.$$

The solution will often be rendered more simple by particular artifices, the proper application of which is best learnt by experience.

Ex. 2. $x^2+y^2=65$, $xy=28$, to find x and y.

From the second equation, $2xy=56$;

adding this to the first, $x^2+2xy+y^2=121$.

Subtracting it from the first, $x^2-2xy+y^2=9$.

By extracting the square roots, $x+y=\pm 11$.

$$x-y=\pm 3.$$

$$\text{therefore } 2x = \pm 14 \text{ or } \pm 8$$

$$x = \pm 7 \quad \text{or } \pm 4.$$

$$\text{and } y = \pm 4 \text{ or } \pm 7.$$

192. It may sometimes be of use to substitute for one of the unknown quantities, the product of the other and a third unkown quantity.*

Ex. 3. $x^2 + xy = 12$, $xy - 2y^2 = 1$, to find x and y.

$$\text{Let } x = vy, \text{ then } v^2y^2 + vy^2 = 12 \qquad \text{(i)}$$

$$\text{and } vy^2 - 2y^2 = 1. \qquad \text{(ii)}$$

Dividing (i) by (ii), $\dfrac{v^2 + v}{v - 2} = 12.$

$$\therefore \; v^2 - 11v = -24$$

$$\therefore \; v^2 - 11v + \tfrac{121}{4} = \tfrac{121}{4} - 24 = \tfrac{25}{4}$$

$$\therefore \; v - \tfrac{11}{2} = \pm \tfrac{5}{2}.$$

$$\therefore \; v = \tfrac{11}{2} \pm \tfrac{5}{2} = 8 \text{ or } 3.$$

From (ii) $y^2 = \dfrac{1}{v - 2} = \tfrac{1}{6}$ or 1.

$$\therefore \; y = \pm \sqrt{(\tfrac{1}{6})} \text{ or } \pm 1.$$

$$x = vy = \pm 8\sqrt{(\tfrac{1}{6})} \text{ or } \pm 3.$$

Ex. 4. $\dfrac{x^2}{y} + \dfrac{y^2}{x} = 18$, $x + y = 12$; to find x and y.

From the first equation, $x^3 + y^3 = 18xy$,

$$\therefore \; (x + y)(x^2 - xy + y^2) = 18xy.$$

$$12(x^2 - xy + y^2) = 18xy. \qquad \text{(A)}$$

From the second equation

$$x^2 + 2xy + y^2 = 144, \text{ or } x^2 + y^2 = 144 - 2xy$$

* This substitution may be successfully applied whenever the sum of the dimensions of he unknown quantities, in every term of each equation, is the same.

From (A) $\quad 2(144-3xy)=3xy$

$$\therefore\ 9xy=288,\ xy=32.\ \therefore x=\frac{32}{y}.$$

$$\therefore \frac{32}{y}+y=12\ ;\quad \therefore\ y^2-12y+32=0.$$

$$\therefore\ y^2-12y+36=36-32=4$$

$$y-6=\pm 2$$

$$\therefore\ y=6\pm 2=8 \text{ or } 4.$$

$$\therefore\ y=4 \text{ or } 8.$$

Otherwise ;—From the second equation

$$x^3+y^3+3xy(x+y)=12^3.$$
$$18xy+3xy\times 12=12^3$$
$$\therefore\ 54xy=12^3;$$
$$xy=1728\div 54=32;\ \text{etc.}$$

193. The operation may sometimes be shortened by substituting for the unknown quantities, the sum and difference of two others.

This artifice may be used when the unknown quantities in each equation are similarly involved.

Ex. 5. Let $x^4+y^4=82$, $x+y=4$, to find the values of x and y.

Assume $x=v+n$.

$$y=v-n.$$
$$x+y=2v=4\ ;\ \therefore\ v=2,$$
$$x^4+y^4=(v+n)^4+(v-n)^4=82.$$
$$\therefore\ 16+24n^2+n^4=41.$$
$$n^4+24n^2+144=144+25=169$$
$$\therefore\ n^2+12=\pm 13.$$
$$\therefore\ n^2=-12\pm 13=1 \text{ or } -25.$$
$$\therefore\ n=\pm 1 \text{ or } \pm 5\sqrt{(-1)}$$
$$x=2\pm 1=3 \text{ or } 1,$$
$$y=2\mp 1=1 \text{ or } 3,\ \text{etc.}$$

Ex. 6. Let $x^5+y^5=244$, $x+y=4$, to find x and y.

Suppose $x=n+v$

$y=n-v$

Then $x+y=2n=4$. $\therefore\ n=2$.

$x^5+y^5=(2+v)^5+(2-v)^5=244$.

$\therefore\ 2(2^5+10\cdot2^3v^2+5\cdot2\cdot v^4)=244$.

$\therefore\ v^4+8v^2=9$.

$\therefore\ v^4+8v^2+16=9+16=25$.

$v^2+4=\pm5$.

$v^2=-4\pm5=1$ or -9.

$\therefore\ v=\pm1$.

$\therefore\ x=2\pm1=3$ or 1, etc.

$\therefore\ y=2\mp1=1$ or 3, etc.

Otherwise :—From the second equation,

$x^2+y^2=16-2xy$ and $x^4+y^4=256+2x^2y^2-64xy$.

But $x^5+y^5=(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)=244$.

$\therefore\ 4\{x^4+y^4-xy(x^2+y^2)+x^2y^2\}=244$.

$256+2x^2y^2-64xy-xy(16-2xy)+x^2y^2=61$.

$\therefore\ x^2y^2-16xy=-39$.

$\therefore\ x^2y^2-16xy+64=64-39=25$.

$\therefore\ xy-8=\pm5$.

$xy=8\pm5=13$ or 3 $\qquad\therefore\ y=\frac{13}{x}$ or $\frac{3}{x}$.

$x+\frac{3}{x}=4,\ x^2-4x+3=0$.	$x^2-4x+13=0$.
$x^2-4x+4=4-3=1$.	$x^2-4x+4=-9$.
$x-2=\pm1$.	$x-2=\pm3\sqrt{(-1)}$.
$x=2\pm1=3$ or 1.	$x=2\pm3\sqrt{(-1)}$.
$y=1$ or 3.	$y=2\mp3\sqrt{(-1)}$.

Another method :—Divide the first equation by the second,

then $x^4-x^3y+x^2y^2-xy^3+y^4=61$. (1)

From the second, $x^4+4x^3y+6x^2y^2+4xy^3+y^4=256$. (2)

Take (1) from (2); $5x^3y+5x^2y^2+5xy^3=195$;

$xy(x^2+xy+y^2)=39$.

But $x^2+2xy+y^2=16$; $\therefore\ x^2+xy+y^2=16-xy$.

$\therefore\ xy(16-xy)=39$ or $x^2y^2-16xy=-39$; etc.

Ex. 7. Let $x^2+2xy=15$ and $xy+2y^2=5$, to find x and y,

$$x^2+2xy=15.$$

From the 2*nd* equation, $2xy+4y^2=10$.

$$\therefore\ x^2+4xy+4y^2=25.$$

$$\therefore\ x+2y=\pm 5.$$

From the first equation $x(x+2y)=15$; $\therefore\ x(\pm 5)=15$.

$$\therefore\ x=\pm 3$$

From the second equation $y(x+2y)=5$; $\therefore\ y=\pm 1$.

Ex. 8. $y^2+x^2+\dfrac{x^4}{y^2}=21$, $y+x+\dfrac{x^2}{y}=7$, to find x and y.

Dividing each side of the first equation by each side of the second, we have

$$y-x+\frac{x^2}{y}=3.$$

Subtracting this from the second equation we have

$$2x=4;\ \therefore\ x=2.$$

Also $y-2+\dfrac{4}{y}=3$; $\therefore\ y^2-5y+4=0$.

Hence $y=4$ or 1.

Ex. 9. $x^3+y^3=9$ and $x+y=3$, to find x and y.

Divide the first equation by the second;

then $x^2-xy+y^2=3$ (1)

But $x^2+2xy+y^2=9$ (2), by squaring 2nd.

Subtract (1) from (2); $\therefore\ 3xy=6$; $\therefore\ xy=2$. (3)

Subtract (3) from 1; $\therefore\ x^2-2xy+y^2=1$.

$$\therefore\ x-y=\pm 1 \text{ and } x+y=3.$$

$\therefore\ 2x=4$ or 2, $2y=2$ or 4; $x=2$ or 1 and $y=1$ or 2.

Ex. 10. $(x-y)(x^2-y^2)=5$, $(x+y)(x^2+y^2)=65$,

to find x and y.

$$x^3-x^2y-xy^2+y^3=5. \quad (1)$$

$$x^3+x^2y+xy^2+y^3=65. \quad (2)$$

Take (1) from (2); $\therefore\ 2x^2y+2xy^2=60$. (3)

Add (3) to (2); $\therefore\ x^3+3x^2y+3xy^2+y^3=125$.

Extract the cube root; $\therefore\ x+y=5$.

From (3), $xy(x+y)=30$. $\therefore\ xy=6$. $\therefore\ y=\frac{6}{x}$.

$\therefore\ x+\frac{6}{x}=5$; $\therefore\ x^2-5x+6=0$; $\therefore\ x=3$ or 2.

$\therefore\ y=2$ or 3.

Ex. 11. $x+y+\sqrt{(x+y)}=6$ and $xy=3$, to find x and y.

$(x+y)+\sqrt{(x+y)}+\frac{1}{4}=6\frac{1}{4}=\frac{25}{4}$.

$\therefore\ \sqrt{(x+y)}+\frac{1}{2}=\pm\frac{5}{2}$.

$\therefore\ \sqrt{(x+y)}=-\frac{1}{2}\pm\frac{5}{2}=2$ or -3.

$x+y=4$ or 9.

$\therefore\ x=4-y$ or $9-y$.

$\therefore\ y(9-y)=3$ or $y(4-y)=3$.

$y^2-9y=-3$ or $y^2-4y=-3$.

$y^2-9y+(\frac{9}{2})^2=\frac{81}{4}-3$ or $y^2-4y+4=4-3$.

$\therefore\ y-\frac{9}{2}=\frac{1}{2}\sqrt{(69)}$ or $y-2=\pm 1$,

$\therefore\ y=2\pm 1=3$ or 1, or $\frac{9}{2}\pm\frac{1}{2}\sqrt{(69)}$.

$\therefore\ x=1$ or 3 etc.

Ex. 12. $\dfrac{x^2}{y^2}+\dfrac{16y^2}{x^2}+\dfrac{x}{y}-\dfrac{4y}{x}=20$, and $x+2y=3$,

to find x and y.

$$\frac{x^2}{y^2}-8+\frac{16y^2}{x^2}+\left(\frac{x}{y}-\frac{4y}{x}\right)+\frac{1}{4}=20-8+\frac{1}{4}=12\frac{1}{4}.$$

$$\therefore\ \left(\frac{x}{y}-\frac{4y}{x}\right)^2+\left(\frac{x}{y}-\frac{4y}{x}\right)+\frac{1}{4}=\frac{49}{4};$$

$$\therefore\ \frac{x}{y}-\frac{4y}{x}+\frac{1}{2}=\pm\frac{7}{2}.$$

$$\therefore\ \frac{x}{y}-\frac{4y}{x}=-\frac{1}{2}\pm\frac{7}{2}=3 \text{ or } -4.$$

$\therefore\ x^2-4y^2=3xy$ or $-4xy$.

$(3-2y)^2-4y^2=3y(3-2y)$ or $-4y(3-2y)$;

$\therefore\ 9-12y+4y^2-4y^2=9y-6y^2$ or $-12y+8y^2$,

$6y^2-21y=-9$, or $8y^2=9$.

$\therefore\ y^2-\frac{7}{2}y+\frac{49}{16}=-\frac{3}{2}+\frac{49}{16}=\frac{25}{16}$.

$\therefore\ y-\frac{7}{4}=\pm\frac{5}{4}$; $\therefore\ y=\frac{7}{4}\pm\frac{5}{4}=3$ or $\frac{1}{2}$.

$x=3-2y=3-2\times 3$ or $3-2\times\frac{1}{2}=-3$ or 2.

From $8y^2=9$, $y=\pm\frac{3}{2}\sqrt{(\frac{1}{2})}$ and $x=3-2y=3\mp 3\sqrt{(\frac{1}{2})}$.

Ex. 13. $x^{y+8}=y^{3x}$, $\sqrt{y}=x$, to find x and y.

$$x=y^{\frac{3x}{y+8}}=y^{\frac{1}{2}};\qquad \therefore \frac{3x}{y+8}=\tfrac{1}{2};\ \therefore y+8=6x.$$

But $y=x^2$; $\therefore x^2-6x+8=0$.

Hence $x=2$ or 4 and $y=4$ or 16.

Ex. 14. Let $x^2+y^2-3x-3y+2xy=0$ and $x+y=5-xy$, to find the values of x and y.

From the 2nd equation $x^2+y^2+2xy=25-10xy+x^2y^2$;

$$\therefore x^2+y^2=25-12xy+x^2y^2.$$

From the 1st equation,

$$(x^2+y^2)-3(x+y)+2xy=0.$$
$$25-12xy+x^2y^2-3(5-xy)+2xy=0.$$
$$x^2y^2-7xy=-10.$$
$$x^2y^2-7xy+(\tfrac{7}{2})^2=-10+\tfrac{49}{4}=\tfrac{9}{4}.$$
$$\therefore xy-\tfrac{7}{2}=\pm\tfrac{3}{2}.$$
$$\therefore xy=\tfrac{7}{2}\pm\tfrac{3}{2}=2 \text{ or } 5.$$
$$x+\tfrac{2}{x}=5-2 \text{ or } x+\tfrac{5}{x}=5-5.$$

$x^2-3x+2=0.$	$x^2+5=0.$
$\therefore x=2$ or 1.	$\therefore x=\pm\sqrt{(-5)}$.
$\therefore y=1$ or 2.	$\therefore y=\mp\sqrt{(-5)}$.

Ex. 15. To solve $\dfrac{x^2}{2y}+\dfrac{x}{2}+y=\sqrt{\left(\dfrac{x^3}{y}+2x^2\right)}$ (A)

$$\tfrac{1}{3}xy+2\sqrt{(x^2-xy+7)}=\tfrac{1}{3}(x^2+16). \qquad \text{(B)}$$

$$\frac{x^4}{4y^2}+\frac{x^2}{4}+y^2+\frac{x^3}{2y}+x^2+xy=\frac{x^3}{y}+2x^2.$$

$$\frac{x^4}{4y^2}-\frac{3x^2}{4}+y^2-\frac{x^3}{2y}+xy=0.$$

$$\therefore \frac{x^2}{4y^2}-\frac{3}{4}+\frac{y^2}{x^2}-\frac{x}{2y}+\frac{y}{x}=0.$$

$$\frac{x^2}{4y^2}-1+\frac{y^2}{x^2}-\left(\frac{xy}{2y}-\frac{y}{x}\right)+\tfrac{1}{4}=0.$$

$$\therefore \left(\frac{x}{2y}-\frac{y}{x}\right)^2-\left(\frac{x}{2y}-\frac{y}{x}\right)+\tfrac{1}{4}=0.$$

$$\therefore \frac{x}{2y} - \frac{y}{x} - \tfrac{1}{2} = 0; \quad \therefore x^2 - 2y^2 - xy = 0.$$

From (B) $(x^2 - xy + 7) - 6\sqrt{(x^2 - xy + 7)} + 9 = 0.$

$$\therefore \sqrt{(x^2 - xy + 7)} - 3 = 0; \quad \therefore x^2 - xy + 7 = 9.$$

$$\therefore x^2 - xy - 2 = 0 = x^2 - 2y^2 - xy.$$

$$\therefore y^2 = 1; \ y = \pm 1.$$

$$x^2 - x(\pm 1) - 2 = 0.$$

$$\therefore x^2 \mp x + \tfrac{1}{4} = \tfrac{9}{4}.$$

$$\therefore x \mp \tfrac{1}{2} = \pm\tfrac{3}{2}; \quad \therefore x = \pm\tfrac{1}{2} \pm \tfrac{3}{2} = \pm 2 \text{ or } \pm 1.$$

Ex. 16. To solve

$$x^3\{y - \sqrt{(y^2 - x^2)}\} = 16\{\sqrt{(xy + x^2)} + \sqrt{(xy - x^2)}\}. \qquad \text{(A)}$$

$$\sqrt{(y + x)} + \sqrt{(y - x)} = 2\sqrt{x}. \qquad \text{(B)}$$

From (A) $xy - \sqrt{(x^2y^2 - x^4)} = \dfrac{16}{x^2} \times \dfrac{2x^2}{\sqrt{(xy + x^2)} - \sqrt{(xy - x^2)}}.$

$$\therefore 2xy - 2\sqrt{(x^2y^2 - x^4)} = \frac{64}{\sqrt{(xy + x^2)} - \sqrt{(xy - x^2)}}.$$

$$\{\sqrt{(xy + x^2)} - \sqrt{(xy - x^2)}\}^2 = \frac{64}{\sqrt{(xy + x^2)} - \sqrt{(xy - x^2)}}.$$

$$\therefore \{\sqrt{(xy + x^2)} - \sqrt{(xy - x^2)}\}^3 = 64.$$

$$\therefore \sqrt{(xy + x^2)} - \sqrt{(xy - x^2)} = 4.$$

$$\therefore 2xy - 2\sqrt{(x^2y^2 - x^4)} = 16.$$

$$\therefore xy - 8 = \sqrt{(x^2y^2 - x^4)}.$$

$$x^2y^2 - 16xy + 64 = x^2y^2 - x^4.$$

$$\therefore x^4 - 16xy + 64 = 0. \qquad (1)$$

From (B) $2y + 2\sqrt{(y^2 - x^2)} = 4x.$

$$\therefore \sqrt{(y^2 - x^2)} = 2x - y.$$

$$y^2 - x^2 = 4x^2 - 4xy + y^2.$$

$$5x^2 = 4xy. \quad \therefore x = 0.$$

$$5x = 4y. \quad \therefore y = \tfrac{5}{4}x.$$

Substitute this value of y in (1), then

$$x^4 - 20x^2 + 64 = 0.$$

Hence $x = \pm 4$ or ± 2, and $y = \tfrac{5}{4}x = \pm 5$ or $\pm\tfrac{5}{2}$.

Ex. 17. Let $x^4=6x+2y$ and $y^4=2x+6y$, to find x and y.

$$x^4=6x+2y,$$
$$y^4=2x+6y.$$
$$\therefore\ x^4+y^4=8(x+y),$$
$$x^4-y^4=4(x-y).$$
$$\therefore\ 4(x-y)(x^4+y^4)=8(x+y)(x^4-y^4).$$
$$\therefore\ 4x^5+12x^4y-12xy^4-4y^5=0.$$
$$\therefore\ x^5+3x^4y-3xy^4-y^5=0.$$

put $x=vy$, and divide by y^5,

$$v^5-1+3(v^4-v)=0.$$
$$\therefore\ v-1=0,\ \therefore\ v=1.$$
$$x=vy=y=\sqrt[3]{(6+2)}=\sqrt[3]{8}=2.$$

Also $v^4+v^3+v^2+v+1+3(v^3+v^2+v)=0.$

$$\therefore\ v^4+4v^3+4v^2+4v+1=0.$$
$$\therefore\ v^2+4v+4+\frac{4}{v}+\frac{1}{v^2}=0.$$
$$\left(v^2+\frac{1}{v^2}\right)+4\left(v+\frac{1}{v}\right)+4=0.$$
$$\therefore\ \left(v+\frac{1}{v}\right)^2+4\left(v+\frac{1}{v}\right)+4=2;\ \text{etc.}$$

Ex. 18. Let $xy+xz=5$, $xy+yz=8$, $xz+yz=9$; to find x, y, z.

Subtract the third equation from the sum of the first and second,

Then $2xy=4$ and $xy=2$. (1)
From the first equation $xz=3$ (2).
From the second $yz=6$ (3)
Divide the product of (1) and (2) by (3),
then $x^2=1$, $x=\pm 1$.
Divide the product of (1) and (3) by (2),
then $y^2=4$, $y=\pm 2$.
Divide the product of (2) and (3) by 1,
then $z^2=9$, $z=\pm 3$.

Ex. 19. Let $\left.\begin{array}{ll} x+y+z=6 & (1) \\ x^2-y^2+z^2=2y+2 & (2) \\ (y^2+2xz)(x+z)-y(x^2+z^2)=6y^2-4 & (3) \end{array}\right\}$ to find x, y, z.

Multiply the first equation by the second and add the third equation to the product, then

$$x^3-y^3+z^3+3x^2z+3xz^2=6y^2+12y+8.$$

$$\therefore\ (x+z)^3=(y+2)^3.$$

$$\therefore\ x+z=y+2.$$

From (1) $x+z=-y+6$.

$$\therefore\ y=2.$$

From (1) $x+z=4$. $\therefore\ x^2+2xz+z^2=16$.

From (2) $x^2+z^2=10$. (A)

$$\therefore\ 2xz=6 \qquad \text{(B)}$$

Subtract (B) from (A), then $x^2-2xz+z^2=4$;

$$\therefore\ x-z=\pm 2.$$

But $x+z=4$.

$$\therefore\ 2x=4\pm 2=6 \text{ or } 2$$

$$\therefore\ x=3 \text{ or } 1.$$

Also $2z=4\mp 2=2$ or 6.

$$\therefore\ z=1 \text{ or } 3.$$

Ex. 20. Let $\frac{8}{5}(x^2+y^2)=\frac{9}{5}(y^2+z^2)=\frac{72}{13}(x^2+z^2)=xyz$, to find x, y, z.

$$\frac{x}{yz}+\frac{y}{xz}=\frac{5}{8} \qquad \text{(A)}$$

$$\frac{y}{xz}+\frac{z}{xy}=\frac{5}{9} \qquad \text{(B)}$$

$$\frac{x}{yz}+\frac{z}{xy}=\frac{13}{72} \qquad \text{(C)}$$

Subtract (C) from the sum of (A) and (B), then

$$\frac{2y}{xz}=1;\qquad \therefore\ \frac{y}{xz}=\frac{1}{2} \qquad (1)$$

From (A) $\qquad \frac{x}{yz}=\frac{1}{8} \qquad (2)$

From (C) $\qquad \frac{z}{xy}=\frac{1}{18} \qquad (3)$

Multiply (1) by (2), then $\frac{1}{z^2}=\frac{1}{16}$; $\therefore \frac{1}{z}=\pm\frac{1}{4}$; $\therefore z=\pm 4$.

Multiply (1) by (3), then $\frac{1}{x^2}=\frac{1}{36}$; $\therefore \frac{1}{x}=\pm\frac{1}{6}$; $\therefore x=\pm 6$.

Multiply (2) by (3), then $\frac{1}{y^2}=\frac{1}{144}$; $\therefore \frac{1}{y}=\pm\frac{1}{12}$; $\therefore y=\pm 12$.

EXAMPLES LXVIII.

Solve the following equations.

1. $3x^2+2y^2=35$, $4x^2-3y^2=24$.
2. $x+y=7$, $xy=12$.
3. $x-y=5$, $xy=36$.
4. $x^2+y^2=25$, $x+y=7$
5. $x^2+y^2=13$, $3x+4y=17$.
6. $x^2-y^2=9$, $x+2y=13$.
7. $x^2+xy=28$, $xy+2y^2=30$.
8. $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}$, $x+y=5$.
9. $\frac{2}{x}+\frac{3}{y}=2$, $x+y=5$.
10. $2x+y=17$, $x^2-y^2=11$.
11. $x^2+xy=45$, $x^2-y^2=9$.
12. $4x^2+7y^2=43$, $3x^2-y^2=26$.
13. $x^2+3xy=70$, $y^2+6=xy$.
14. $x^2+xy+3y^2=93$, $2x^2-3xy+y^2=6$.
15. $x^2-2xy+2y^2=10$, $2x^2-4xy+y^2=-7$.
16. $x^2+2xy=45$, $xy+3y^2=22$.
17. $x^2-xy=15$, $y^2-xy=10$.
18. $x^2-9xy=10$, $x+y=11$.
19. $x^2-9y^2=16$, $xy-3y^2=2$.
20. $x^2+4xy+y^2=121$, $5x^2+2y^2=130$.
21. $(x+3y)(3x+y)=60$, $2x(x+y)=25-y^2$.
22. $x^2+xy+y^2=208$, $x+y=16$.
23. $4x^2-7y^2+27=0$, $x+y=6$.
24. $x^2-3xy+2y^2=3$, $2x+3y=16$.

25. $x^2-3xy-10=0,\ xy-3y^2=2.$

26. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=4\frac{1}{4},\ x^2-y^2=9.$

27. $x+\dfrac{1}{y}=5\frac{1}{2},\ y+\dfrac{1}{x}=2\frac{1}{5}.$ 28. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{6},\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{13}{36}.$

29. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\frac{5}{2},\ x^2+y^2=90.$

30. $2x+y=(x-1)^2,\quad x-y=3.$

31. $x^2+y^2+x+y=32.\quad x^2-y^2+x-y=8.$

32. $x+y+\sqrt{(xy)}=28,\ x^2+y^2+xy=336.$

33. $y^2=y+x,\ x^2=5x+2y.$

34. $20x^2-12y^2=x^2y^2,\ 4y^2-3x^2=7.$

35. $x^2=ax+by,\ y^2=bx+ay.$

36. $(x+2y)^2-x=2(y+6),\ 2x+y=5.$

37. $y^2-4xy+9x^2+2x=6y.\quad 3y^2-12xy+2x^2-4x+12y=0$

38. $y^3-x^3=19,\ y-x=1.$ 39. $x^3+y^3=28,\ x+y=4.$

40. $3(x^2-1)=4xy,\ 2(2y^2+1)=3xy.$

41. $ax-by=mxy,\ \dfrac{a}{x}-\dfrac{b}{y}=n.$

42. $2y^2=xy+2,\ 4x^2=xy+30.$

43. $xy+y=9,\ x^2y^2+y^2=45.$ 44. $\sqrt{x}-\sqrt{y}=1,\ \sqrt{(xy)}=12.$

45. $\dfrac{1}{x}+\dfrac{1}{y}=\frac{3}{4};\qquad \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{5}{16}.$

46. $x^3+y^3=28,\ x^2y+xy^2=12.$

47. $\dfrac{x^2}{y}-\dfrac{y^2}{x}=7,\ x-y=2.$

48. $(x+y)(x^2+y^2)=175,\ (x-y)(x^2-y^2)=7.$

49. $x^2+y^2+\sqrt{(x^2+y^2)}=30,\ x+y=\frac{7}{12}xy.$

50. $x^2+y^2=136,\ x^2-y^2+\sqrt{(x^2-y^2)}=72.$

51. $\frac{1}{x}+\frac{1}{y}=\frac{6}{x^2y^2},\ x+y=\frac{3}{2}xy.$

52. $x-y=7,\ \sqrt[3]{x}-\sqrt[3]{y}=1.$

53. $a^x=a^5a^y,\ a^{\frac{x}{2}}=a^{\frac{6}{y}}a^{\frac{x}{3}}.$ 54. $x^{x+y}=y^4;\ y^{x+y}=x.$

55. $x^my^n=a^nb^mc^{m+n},\ x^ny^m=a^mb^nc^{m+n}$; m and n are not both odd or both even.

56. $\sqrt{(y+x)}+\sqrt{(y-x)}=2x,\ 5x=3y.$

57. $\frac{1}{x}+\frac{1}{y}=\frac{9}{20},\ \frac{1}{x+1}+\frac{1}{y+1}=\frac{11}{30}.$

58. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=2\frac{1}{20};\ \sqrt{\frac{x^2}{y}}+\sqrt{\frac{y^2}{x}}=9\frac{9}{20}.$

59. $x^4+y^4-x^2-y^2=12,\ x^2+y^2+x^2y^2=9.$

60. $\frac{x^2}{y^2}+\frac{4x}{y}=\frac{85}{9},\ x-y=2.$ 61. $xy+\frac{x}{y}=\frac{5}{3},\frac{1}{xy}+\frac{y}{x}=\frac{20}{3}.$

62. $x^4=3x+2y,\ y^4=2x+3y.$

63. $xy=(x-\frac{3}{7})(y+\frac{2}{3}),\ x^2y^2=(x^2+3)(y^2-4).$

64. $x^3+11xy^2=30,\ yx^2+y^3=5.$

65. $x+2=2(x-y)=2\sqrt{(x+6y-1)}.$

66. $2\sqrt{x}+3\sqrt{y}=13=\frac{1}{2}(x+y).$

67. $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{x}+\frac{1}{y}=\frac{43}{36},\ xy=6.$

68. $b^2x^2+a^2y^2=a^2b^2,\ ax-by=0.$

69. $x^4+y^4=272,\ x-y=2.$ 70. $x^5-y^5=31,\ x-y=1.$

71. $xy(x^2+y^2)=10,\ x^2y^2(x^4+y^4)=68.$

72. $x^2+xy+y^2=7,\ x^4+x^2y^2+y^4=21.$

73. $x^2+xy+y^2=\frac{21}{16},\ x+\sqrt{(xy)}+y=\frac{7}{4}.$

74. $x-\sqrt{x}=7-y,\ y-\sqrt{y}=9-x.$

75. $x+\sqrt{(x+y)}=6-y,\ x^2+y^2=10.$

76. $x^5-y^5=33a^5,\ x-y=a.$

77. $x^3+18xy^2+18y^3=99,\ y(x^2+xy+y^2)=13.$

78. $x^4+10x^2y^2+y^4=172,\ xy(x^2-xy+y^2)=21.$

79. $x^5+y^5+5x^2y^2(x+y)=93,\ xy(x^2+y^2)(x+y)=30.$

80. $(x+y)(x^2+y^2)=15,\ (x^2+y^2)(x^4+y^4)=85.$

81. $3x+y+4\sqrt{(3x+y+6)}=26,\ 3x^2-8x=y^2+2y.$

82. $\sqrt{(x^2-2y^2)}-2\sqrt{x}+\dfrac{x}{\sqrt{(x^2-2y^2)}}=0,\ x+y=3.$

83. $x^2+3x+\dfrac{16y^4}{x^2+3x}=8y^2,\ x+y=2.$

84. $\dfrac{x+\sqrt{(x^2-4y^2)}}{x-\sqrt{(x^2-4y^2)}}=64\dfrac{\sqrt{(x+2y)}-\sqrt{(x-2y)}}{\sqrt{(x+2y)}+\sqrt{(x-2y)}};\ \ x^2-y^2=68\frac{1}{4}.$

85. $\dfrac{x+\sqrt{(x^2-y^2)}}{x-\sqrt{(x^2-y^2)}}+\dfrac{x-\sqrt{(x^2-y^2)}}{x+\sqrt{(x^2-y^2)}}=9\frac{1}{9},$

$$2x^2+2+\sqrt{(x^2-5xy+6)}=10xy.$$

86. $\sqrt{\{(x^2-1)(y^2-1)\}}=1=\frac{12}{25}\sqrt{(x^2+y^2)}.$

87. $x\sqrt{(y^2-1)}-y\sqrt{(x^2-1)}=\frac{1}{5}\sqrt{(x^2+y^2)}=\frac{5}{12}.$

88. $x(y+z)=24,\ y(x+z)=40,\ z(x+y)=48.$

89. $x^2+y^2+z=17,\ y^2+7=2xz,\ z=2x.$

90. $x+y+z=7,\ x^2+y^2+z^2=21,\ xz=y^2.$

91. $x+2y+z=8,\ x^2+4y^2+z^2=26,\ xy=2.$

92. $x+y+z=6,\ xy+xz+yz=11,\ xyz=6.$

93. $x^2+y^2+z^2=50,\ y^2+14=2xz,\ y^2=xz.$

94. $x^2+y^2+z^2=29,\ y^2+x=11,\ y^2+z=13.$

95. $y+xz=z+xy=5,\ x+yz=7\frac{1}{4}.$

96. $x^2+xy+xz=6,\ y^2+xy+yz=12,\ z^2+zy+xz=18.$

97. $x^2y^2=8z,\ x^2z^2=64y,\ y^2z^2=1024.$

98. $x^my^n=a^mc^nb^{m+n},\ x^mz^r=b^mc^ra^{n+r},\ y^nz^r=b^na^rc^{n+r}.$

99. $x^y = y^{\frac{y+2}{2}}, \; y^{3y} = z^2, \; x^3 = z.$

100. $\dfrac{xyz}{x+y} = 2, \; \dfrac{xyz}{x+z} = \frac{3}{2}, \; \dfrac{xyz}{y+z} = \frac{6}{5}.$

101. $4y + 2z + \dfrac{4}{x} = 7, \; 4z + y + \dfrac{1}{y} = 4, \; \frac{1}{4}x + \frac{1}{2}y = 1.$

102. $$\left.\begin{aligned} 3x + 3y - z = 3, \quad x^2 + y^2 - z^2 &= \frac{14 - 9z}{2}, \\ x^3 + y^3 + z^3 = 3xyz &+ \frac{17z + 44}{4}. \end{aligned}\right\}$$

103. $(x-2)^2 + (y-3)^2 + (z-1)^2 = 24, \; xy + yz + xz = 63,$
$2x + 3y + z = 30.$

104. $x^2 + y^2 - z^2 = (x + y - z)^2 - 4,$
$x^3 + y^3 - z^3 = (x + y - z)^3 - 30, \; xy = 6.$

105. $$\left.\begin{aligned} (x + yz)(y + xz)(z + xy) &= 1540. \\ (x^2 - 1)(y^2 - 1)(z^2 - 1) &= 360. \\ x^2 + y^2 + z^2 + 2xyz &= 77. \end{aligned}\right\}$$

Theory of Quadratic Equations and Quadratic Expressions.

194. *A quadratic equation, properly so called, cannot have more than two different values of the unknown quantity which will satisfy it, or it cannot have more than two different roots.*

Let $ax^2 + bx + c = 0$.

By solving the equation we obtain the following two values of x:—

$$\frac{-b + \sqrt{(b^2 - 4ac)}}{2a} \text{ and } \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}.$$

If α and β represent these values, then α and β are the only two values of x which satisfy the equation.

If possible, let γ represent a third value of x different from α and β. By successively substituting these values in the original equation, we have,

$$a\alpha^2+b\alpha+c=0 \quad \dots \quad \dots \quad (A)$$
$$a\beta^2+b\beta+c=0 \quad \dots \quad \dots \quad (B)$$
$$a\gamma^2+b\gamma+c=0 \quad \dots \quad \dots \quad (C)$$

Subtracting (B) from (A) $a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$.

Dividing by $\alpha-\beta$, which by supposition is not zero, we have

$$a(\alpha+\beta)+b=0. \qquad (1)$$

Subtract (C) from (A) and divide by $\alpha-\gamma$,

$$\text{then, } a(\alpha+\gamma)+b=0 \qquad (2)$$

Subtract (2) from (1), $a(\beta-\gamma)=0$.

Now a is not zero, because then the equation will become $bx+c=0$, a simple equation, therefore $\beta-\gamma=0$; that is, $\beta=\gamma$, which is impossible, because by supposition γ is different from β. Hence a quadratic equation cannot have more than two different roots.

195. By dividing any equation by the coefficient of the second power of the unknown quantity, the equation can be reduced to the form $x^2+px+q=0$.

196. *In a quadratic equation of the form $x^2+px+q=0$, the sum of the roots is equal to the coefficient of the second term with its sign changed, and the product of the roots is equal to the last term.*

By solving the equation, the two roots we get are

$$\frac{-p+\sqrt{(p^2-4q)}}{2} \text{ and } \frac{-p-\sqrt{(p^2-4q)}}{2}.$$

If α and β represent the two roots, then

$$\alpha+\beta=-p.$$

$$\alpha\beta=\frac{p^2-p^2+4q}{4}=q.$$

Obs. We may now easily find the values of expressions in which α and β occur symetrically.

Ex. 1. $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=p^2-2q.$

Ex. 2. $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=p^2-4q.$

Ex. 3. $\dfrac{1}{\alpha}+\dfrac{1}{\beta}=\dfrac{\alpha+\beta}{\alpha\beta}=\dfrac{-p}{q}=-\dfrac{p}{q}.$

Cor. If we know one root of a quadratic equation, we may find the other without solving the equation.

Ex. $\dfrac{x^2}{p-q}+\dfrac{m}{p+q}=\dfrac{mx}{p-q}+\dfrac{x}{p+q}$.

Here $x=m$ obviously satisfies the equation; multiplying by $p-q$ and transposing, we get

$$x^2-\left(m+\frac{p-q}{p+q}\right)x+\frac{m(p-q)}{p+q}=0.$$

Hence $\dfrac{p-q}{p+q}$ is the other value of x.

197. If α and β be the roots of the equation $x^2+px+q=0$, then the expression $x^2+px+q=(x-\alpha)(x-\beta)$.

Because $p=-(\alpha+\beta)$ and $q=\alpha\beta$.

$$\begin{aligned}\therefore\ x^2+px+q&=x^2-(\alpha+\beta)x+\alpha\beta\\&=(x-\alpha)(x-\beta).\end{aligned}$$

Cor. Hence we may form an equation of which the roots are given.

Ex. 1. To form the equation whose roots are 7 and 8.

$x-\alpha=x-7$ and $x-\beta=x-8$.

$\therefore$ the required equation is

$(x-7)(x-8)=0$, or $x^2-15x+56=0$

Ex. 2. To form the equation whose roots are 8 and -10.

$x-\alpha=x-8$ and $x-\beta=x+10$.

$\therefore$ $(x-8)(x+10)=0$, or $x^2+2x-80=0$.

198. *The two roots of the equation* $x^2+px+q=0$ *are real and equal, real and different or impossible according as* p^2 *is* $=$, $>$ *or* $<4q$, *and the converse.*

The roots are $\frac{1}{2}\{-p+\sqrt{(p^2-4q)}\}$ or $\frac{1}{2}\{-p-\sqrt{(p^2-4q)}\}$

First. Let $p^2=4q$, then $\sqrt{(p^2-4q)}$ is equal to zero and then each of the roots is equal to the real quantity $\dfrac{-p}{2}$.

Secondly. Let $p^2>4q$, then $\sqrt{(p^2-4q)}$ is a *real* quantity and the roots are different in value.

Lastly. Let $p^2<4q$, then $\sqrt{(p^2-4q)}$ is an *imaginary* quantity and the roots are both impossible.

199. If p^2-4q be equal to zero, then the left hand expression of the equation $x^2+px+q=0$ is a perfect square, and the converse.

For in this case the two roots α and β are equal.

$$\therefore \quad x^2+px+q=x^2-(\alpha+\beta)x+\alpha\beta=x^2-2\alpha x+\alpha^2=(x-\alpha)^2.$$

COR. Hence the expression x^2+px+q is a perfect square if the two roots of the equation $x^2+px+q=0$ are equal, and the converse.

200. In the equation $ax^2+bx+c=0$, if $b^2>4ac$ and if b^2-4ac be a perfect square, then the roots of the equation are *rational* and the expression ax^2+bx+c can be resolved into rational factors, and the converse.

The roots are $\dfrac{-b+\sqrt{(b^2-4ac)}}{2a}$ and $\dfrac{-b-\sqrt{(b^2-4ac)}}{2a}$.

They are rational when b^2-4ac is a perfect square. Let α and β represent the two roots, then dividing the equation by a

$$x^2+\frac{b}{a}x+\frac{c}{a}=x^2-(\alpha+\beta)x+\alpha\beta=(x-\alpha)(x-\beta)=0.$$

$$\therefore \quad ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a(x-\alpha)(x-\beta).$$

COR. The expression ax^2+bx+c can be resolved into rational factors if the two roots of the equation $ax^2+bx+c=0$ are rational and different, or in other words, when $b^2>4ac$ and b^2-4ac is a perfect square.

OBS. The student should be careful to distinguish between a *quadratic expression* and a *quadratic equation*. In the quadratic equation $ax^2+bx+c=0$, x may have only *two distinct* values, but in a quadratic expression ax^2+bx+c, when it is not made equal to zero, x may have any value whatever.

Ex. 1. Resolve $x^2-13x+40$ into simple factors.

Suppose $x^2-13x+40=0$.

Hence $x=5$ or 8.

$$\therefore \quad x^2-13x+40=(x-\alpha)(x-\beta)=(x-5)(x-8)=0.$$

$\therefore$ $x-5$ and $x-8$ are the two factors.

Ex. 2. Resolve $6x^2-13x+6$ into simple factors.

Suppose $6x^2-13x+6=0$;

then $x=\frac{3}{2}$ or $\frac{2}{3}$.

$$\therefore \quad 6x^2-13x+6=6(x^2-\tfrac{13}{6}x+1)=6(x-\tfrac{3}{2})(x-\tfrac{2}{3})$$
$$=(2x-3)(3x-2).$$

Ex. 3. Resolve $x^2-(y+c)x-2y^2+5cy-2c^2$ into simple factors.

Suppose $x^2-(y+c)x-2y^2+5cy-2c^2=0$.

Hence $x=2y-c$ or $-y+2c$.

$\therefore$ $x-2y+c$ and $x+y-2c$ are the two factors.

201. The quadratic expression ax^2+mx+n is divisible by $x-r$ only when r is a root of the equation $ax^2+mx+n=0$.

Divide ax^2+mx+n by $x-r$, the quotient is $ax+ar+m$ and the remainder is ar^2+mr+n. Since by the question there should remain no remainder, the expression ar^2+mr+n must vanish, that is, $ar^2+mr+n=0$, therefore r is a root of the equation $ax^2+mx+n=0$.

202. To calculate the sign and value of the expression ax^2+bx+c.

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right).$$

Let p, q represent $\frac{b}{a}$, $\frac{c}{a}$ respectively; then the expression $=a(x^2+px+q)$. Hence the sign and value of the original expression are ascertained when the sign and value of the expression x^2+px+q are known. Therefore we will calculate the sign and value of the expression x^2+px+q, in which each of p and q is positive or negative, fractional or integral.

FIRST. If $p^2<4q$, the trinomial x^2+px+q can never become negative, whatever *real* value be assigned to x.

For $x^2+px+q=x^2+px+\frac{1}{4}p^2+\frac{1}{4}(4q-p^2)$

$=(x+\frac{1}{2}p)^2+\frac{1}{4}(4q-p^2)$.

Now $(x+\frac{1}{2})^2$ and $\frac{1}{4}(4q-p^2)$ are positive, whatever real value be assigned to x. $\therefore$ the expression x^2+px+q is positive.

COR. If $p^2<4q$, then the roots of the equation $x^2+px+q=0$, must be *imaginary*; therefore the expression x^2+px+q is positive when the roots of the equation $x^2+px+q=0$ are *imaginary*.

SECONDLY. If $p^2=4q$, the trinomial can never become negative, whatever real value be assigned to x.

For $x^2+px+q=x^2+px+\frac{1}{4}p^2+\frac{1}{4}(4q-p^2)$

$=x^2+px+\frac{1}{4}p^2+0$

$=(x+\frac{1}{2}p)^2$, a positive quantity.

COR. Hence the expression x^2+px+q is positive when the roots of the equation $x^2+px+q=0$ are equal.

LASTLY. If $p^2 > 4q$, the expression x^2+px+q may be positive or negative. It is negative when x lies in value between α and β, the two real roots of the equation $x^2+px+q=0$.

$$\text{For, } x^2+px+q=x^2+px+\tfrac{1}{4}p^2-\tfrac{1}{4}p^2+q$$
$$=(x+\tfrac{1}{2}p)^2-\tfrac{1}{4}(p^2-4q).$$

The expression is positive or negative according as $(x+\frac{1}{2}p)^2 >$ or $< \frac{1}{4}(p^2-4q.)$.

$$x^2+px+q=\{x+\tfrac{1}{2}p-\tfrac{1}{2}\sqrt{(p^2-4q)}\}\{x+\tfrac{1}{2}p+\tfrac{1}{2}\sqrt{(p^2-4q)}\}$$
$$=(x-\alpha)(x-\beta),$$

where α and β represent the two real roots of the equation $x^2+px+q=0$, that is, $\alpha=-\frac{1}{2}p+\frac{1}{2}\sqrt{(p^2-4q)}$ and $\beta=-\frac{1}{2}p-\frac{1}{2}\sqrt{(p^2-4q)}$.

The expression $(x-\alpha)(x-\beta)$ is negative when one of the factors $x-\alpha$ and $x-\beta$ is positive and the other negative. Then x must lie in value between α and β.

Suppose $x-\alpha$ be positive and $x-\beta$ negative, then x must be greater than α and less than β. Therefore x lies in value between α and β.

Likewise, if $x-\alpha$ be negative and $x-\beta$ positive, then x would be less than α and greater than β.

Therefore the expression $(x-\alpha)(x-\beta)$ is negative only when x lies in value between α and β.

Wherefore we conclude that the expression x^2+px+q is always positive and may have any value whatever for all real values of x, except in the case when the roots of the equation $x^2+px+q=0$ are *real* and *different* and x lies in value between them.

Ex. If x be real, to prove that $\dfrac{x^2+10x-35}{2x-6}$ cannot have any real value between 6 and 10.

$$\text{Suppose } \frac{x^2+10x-35}{2x-6}=p, \text{ then}$$
$$x^2+10x-35=2px-6p.$$
$$x^2-2(p-5)x+6p-35=0$$
$$\text{Hence } x=(p-5)\pm\sqrt{(p^2-16p+60)}.$$

If x be real, $p^2-16p+60$ must be positive; that is $(p-6)(p-10)$ must be positive. $\therefore$ p cannot lie between 6 and 10.

Wherefore if x be real, $\dfrac{x^2+10x-35}{2x-6}$ can have no real value between 6 and 10.

203. We will add here some miscellaneous examples on the fore-going articles.

Ex. 1. To find the sum, difference, and the product of the roots of the equation $x^2-50x+576=0$.

$$\alpha+\beta=50,\ \alpha\beta=576.$$
$$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=50^2-4\times24^2=50^2-48^2.$$
$$\therefore\ \alpha-\beta=\sqrt{\{(50+48)(50-48)\}}$$
$$=\sqrt{(98\times2)}=\sqrt{(49\times2\times2)}$$
$$=7\times2=14.$$

Ex. 2. For what value of n will the equation $4x^2-12x+n=0$ have equal roots ?

$$x^2-3x+\tfrac{1}{4}n=0$$

Solving the equation $x=\frac{1}{2}\{3\pm\sqrt{(9-n)}\}$.

If the equation have equal roots the expression $\sqrt{(9-n)}$ must be zero (Art. 198). $\therefore\ 9-n=0$; $\therefore\ n=9$.

Ex. 3. For what value of a will the expression $x^2-(3a-2)x+2a^2-a-3$ be a perfect square? (x is a real quantity).

The equation $x^2-(3a-2)x+2a^2-a-3=0$ will have equal roots (Art. 199).

$$\therefore\ (3a-2)^2=4(2a^2-a-3). \qquad \text{(Art. 198)}$$

Hence $a=4$

Ex. 4. For what values of a will the equation $x^2-(3a-1)x+2a^2+2a-11=0$ have equal roots ? x is a real quantity.

$$(3a-1)^2=4(2a^2+2a-11) \qquad \text{Art. 198.}$$

Hence $a=5$ or 9.

Ex. 5. If α and β be the roots of the equation $ax^2-bx+c=0$, form the equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$.

$$x^2-\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)x+\frac{1}{\alpha\beta}=0$$
$$\therefore\ \alpha\beta x^2-(\alpha+\beta)x+1=0.$$
$$\therefore\ \frac{c}{a}x^2+\frac{b}{a}x+1=0.$$
$$\therefore\ cx^2+bx+a=0.$$

Ex. 6. If α and β be the roots of the equation $x^2+px+q=0$, form a quadratic equation with rational coefficients, one of whose roots is $\alpha\beta+\sqrt{(\alpha+\beta)}$.

The other root must be such a multiplier as will rationalise $\alpha\beta+\sqrt{(\alpha+\beta)}$, also it must be such a quantity which when added to $\alpha\beta+\sqrt{(\alpha+\beta)}$ will give a rational sum.

$$\alpha\beta-\sqrt{(\alpha+\beta)} \text{ is such a quantity.}$$

Suppose $\alpha\beta+\sqrt{(\alpha+\beta)}=R$ and $\alpha\beta-\sqrt{(\alpha+\beta)}=r$.

Then the required equation is

$$(x-R)(x-r)=0.$$

$$\text{or } x^2-(R+r)x+Rr=0.$$

$$\text{or } x^2-2\alpha\beta x+\alpha^2\beta^2-(\alpha+\beta)=0.$$

$$\text{or } x^2-2qx+q^2+p=0.$$

Ex. 7. If the equations $ax^2+bx+c=0$ and $cx^2+ax+b=0$ have a common root, shew that $a^3+b^3+c^3=3abc$.

Let r be the common root, then

$$ar^2+br+c=0 \qquad \text{(A)}$$

$$cr^2+ar+b=0. \qquad \text{(B)}$$

Subtract c times (A) from a times B and transpose, then

$$(a^2-bc)r=c^2-ab. \qquad (1)$$

Subtract c times (B) from b times (A) etc., then

$$b^2-ac=(c^2-ab)r. \qquad (2)$$

Multiply (1) by (2) and divide by r, then

$$(a^2-bc)(b^2-ac)=(c^2-ab)^2.$$

Hence $a^3+b^3+c^3=3abc$.

Ex. 8. If $x-r$ be a common measure of $a^2x^2+acx+c^2$ and $c^2x^2+bcx+b^2$, then shew that $c^4+abc^2+a^2b^2=0$.

r is a common root of $a^2x^2+acx+c^2=0$

and $c^2x^2+bcx+b^2=0$.

$$\left.\begin{array}{ll}\text{Therefore } a^2r^2+acr+c^2=0 & \text{(A)}\\ c^2r^2+bcr+b^2=0 & \text{(B)}\end{array}\right\} \text{Art. 201.}$$

Subtract b^2 times (A) from c^2 times (B) and c^2 times (A) from a^2 times (B), then

$$(c^4-a^2b^2)r=-bc(c^2-ab) \qquad (1)$$

$$\text{and } (c^4-a^2b^2)=-ac(c^2-ab)r \qquad (2)$$

$$\therefore (c^2+ab)^2=abc^2.$$

$$\therefore c^4+abc^2+a^2b^2=0.$$

R

Ex. 9. If $x-a$ be the square root of x^2-nx+m^2, then shew that $(a^2-m^2)^2=a(an-2m^2)(2a-n)$; x is a real quantity.

a is the only root of the equation $x^2-nx+m^2=0$. (1)

$\therefore (x-a)=0$; $\therefore x^2-2ax+a^2=0$. (2)

Subtract (2) from (1) and m^2 times (2) from a^2 times (1) then

$$(2a-n)x=a^2-m^2,$$

$$\text{and } (an-2m^2)a=(a^2-m^2)x.$$

$$\therefore a(2a-n)(an-2m^2)=(a^2-m^2)^2.$$

Ex. 10. For what values of n will the equation $x^2-6xy+6x+8y^2-4ny+8=0$ have rational roots. x, y are real quantities. Then $(6y-6)^2-4(8y^2-4ny+8)$ is a perfect square. Art. 200.

$\therefore y^2+2(2n-9)y+1$ is a perfect square

$\therefore y^2+2(2n-9)y+1=0$ has equal roots. Art. 199.

$\therefore \{2(2n-9)\}^2-4=0$ Art. 198.

$\therefore n=5$ or 4.

Ex. 11. The expression $x^2+axy+4xz+5a^2yz-2a^2y^2-32z^2$ is resolvable into simple factors; find the values of a. (x, y, z are real quantities).

The equation $x^2+axy+4xz+5a^2yz-2a^2y^2-32z^2=0$ has rational roots. Art. 200.

$\therefore (ay+4z)^2-4(5a^2yz-2a^2y^2-32z^2)$ is a perfect square Art. 200.

$\therefore (ay+4z)^2-4(5a^2yz-2a^2y^2-32z^2)=0$ has equal roots Art. 199.

or $9a^2y^2+4a(2z-5az)y+144z^2=0$ has equal roots.

$\therefore 16a^2(2z-5az)^2-4\times 9a^2\times 144z^2=0$. Art. 198.

$\therefore (2-5a)^2-36\times 9=0$.

$\therefore a=4$ or $-\frac{16}{5}$.

Ex. 12. If $m=\dfrac{a+b}{a-b}$, m, a, b being rational quantities, shew that the expression $x^2-mx+\dfrac{2bm}{a+b}$ can be resolved into rational factors.

$$m^2-4\times\frac{2mb}{a+b}=\left(\frac{a+b}{a-b}\right)^2-\frac{8b}{a+b}\times\frac{a+b}{a-b}$$

$$=\left(\frac{a+b}{a-b}\right)^2-\frac{8b}{a-b}=\left(\frac{a-3b}{a-b}\right)^2;$$

hence etc. Art. 200.

Ex. 13. If the expression $ax^2+by^2+cz^2+2ayz+2bxz+2cxy$ be resolvable into rational factors, then shew that $a^3+b^3+c^3=3abc$.

The roots of $ax^2+2(bz+cy)x+by^2+cz^2+2ayz=0$
are rational. Art. 200.

$\therefore$ $\{2(bz+cy)\}^2-4(by^2+cz^2+2ayz)a$ is a perfect square.
Art. 200.

$\therefore$ the roots of $(bz+cy)^2-(by^2+cz^2+2ayz)a=0$
are equal. Art. 199.

$\therefore$ the roots of $(c^2-ab)y^2-2(a^2-bc)zy+b^2z^2-acz^2=0$
are equal.

$\therefore$ $\{2(a^2-bc)z\}^2-4(z^2b^2-acz^2)(c^2-ab)$ must vanish.

$\therefore$ $(a^2-bc)^2-(b^2-ac)(c^2-ab)=0$.

Hence $a^3+b^3+c^3=3abc$.

Ex. 14. Shew that $\dfrac{2x^2+4x+1}{x^2+4x+2}$ will be capable of all value whatever for all real values of x.

$$\text{Let } \frac{2x^2+4x+1}{x^2+4x+2}=y.$$

$$\therefore\ 2x^2+4x+1=yx^2+4yx+2y$$

$$\therefore\ (2-y)x^2+4(1-y)x+(1-2y)=0.$$

$$\therefore\ x=\frac{-4(1-y)\pm\sqrt{\{16(1-y)^2-4(2-y)(1-2y)\}}}{2(2-y)}$$

$\therefore$ the expression $4(1-y)^2-(2-y)(1-2y)$

$=2y^2-3y+2$

$=2(y^2-\frac{3}{2}y+1)=2(y^2-\frac{3}{2}y+\frac{9}{16}+\frac{7}{16})$

$=2\{(y+\frac{3}{4})^2+\frac{7}{16}\}$ is always positive.

Hence y may have any value whatever for all real values of x.

Ex. 15. Shew that for all real values of x the expression $\dfrac{x^2-2x+4}{x^2+2x+4}$ lies between $\frac{1}{3}$ and 3.

$$\text{Let } \frac{x^2-2x+4}{x^2+2x+4}=y.$$

$$x^2-2x+4=x^2y+2xy+4y.$$

$$\therefore\ (1-y)x^2-2(1+y)x+4-4y=0.$$

$$\therefore\ x=\frac{2(1+y)\pm\sqrt{\{4(1+y)^2-16(1-y)^2\}}}{2(1-y)}$$

$$=\frac{1+y\pm\sqrt{\{(1+y)^2-4(1-y)^2\}}}{1-y}.$$

If x be real, the expression $(1+y)^2-4(1-y)^2$

$=(1+y+2-2y)(1+y-2+2y)$

$=(3-y)(3y-1)$ must be positive.

But it is positive only when y lies between 3 and $\frac{1}{3}$. $\therefore$ etc

EXAMPLES LXIX.

1. Construct the quadratic equations whose roots are

(1) 16 and 5 (2) 4 and $\frac{1}{4}$. (3) 8 and $-\frac{1}{4}$.

(4) -4 and -5. (5) $a-b$ and $a+b$

(6) $\sqrt{5}$ and $-\sqrt{5}$. (7) $\sqrt{3}+1$ and $\sqrt{3}-1$.

(8) $\dfrac{a+b}{a-b}$ and $\dfrac{a-b}{a+b}$ (9) $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$ and $\dfrac{\sqrt{5}-1}{\sqrt{5}+1}$.

(10) $\sqrt{(a+b)}+\sqrt{(a-b)}$ and $\sqrt{(a+b)}-\sqrt{(a-b)}$

2. If α and β be the roots of the equation $x^2-7px-12p^2=0$, form the equation whose roots are $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$.

3. If a and b be the roots of the equation $x^2-2px+p^2-q^2=0$, form the equation whose roots are $a+b$ and $a-b$.

4. If $\dfrac{1}{a}$ and $\dfrac{1}{b}$ be the roots of the equation $x^2+px+q-q^2=0$, form the equation whose roots are a^2 and b^2.

5. If α and β be the roots of the equation $2x^2-2(a+b)x+\frac{1}{2}(a^2-ab+b^2)=0$, find the value of $\alpha^2+\beta^2$.

6. Given the equations $x^2-px+r=0$, and $x^2-nx+m=0$, each of the two roots of the first is reciprocal of the two roots of the other; prove that $mp^2=rn^2$.

7. If a and b be the roots of the equation $x^2-50x+600$, find the values of a^2+ab+b^2, a^3+b^3 and $a^4+a^2b^2+b^4$.

8. If a and b be the roots of the equation $x^2-px+q=0$, form the equation whose roots are $\dfrac{a^2}{b}$ and $\dfrac{b^2}{a}$.

9. If a and b be the roots of the equation $x^2+px+q=0$, shew that $\dfrac{a}{b}+\dfrac{b}{a}-\dfrac{p^2}{q}+2=0$.

10. Find the sum and product of the roots of the equation, $\frac{a}{x-a}+\frac{b}{x-b}+\frac{c}{x-c}=0$.

11. Form a quadratic equation with rational coefficients one of whose roots is $a+b+\sqrt{(a^2+b^2)}$.

12. If α and β be the roots of the equation $x^2+bx+c=0$, form a quadratic equation with rational coefficients, one of whose roots is $\alpha\beta+\sqrt{(\alpha^2+\beta^2)}$.

13. Find the value of a^5+b^5, a and b being the roots of the equation $x^2-px+q=0$.

14. If α and β be the roots of the equation $qx^2+qx+1=0$, find the value of $\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}$.

15. If a and b be the roots of the equation $x^2+px+q=0$, form the equation whose roots are a^3+b^3 and a^2+b^2.

16. Find the relation between the co-efficients of the equation $x^2+px+q=0$ one of whose roots is n times the other.

17. Find the value of $a^4+2a^3b+2a^2b^2+2ab^3+b^4$ where a and b are the other two roots of the equation $2x^3-(x-2)^3-10(x-2)^2-26(x-2)=16$, *beside* the root $x=2$.

18. Shew that the roots of the equation $x^2-(p^2-2q)x+q^2=0$, are the squares of the roots of $x^2-px+q=0$.

19. Shew that the roots of the equation $x^2-2(p^2-2q^2)x+p^2(p^2-4q^2)=0$ are the squares of the sum and difference of the roots of the equation $x^2-px+q^2=0$.

20. Resolve the following quadratic expressions into the product of simple factors.

(1) $x^2-43x+280$. (2) $15x^2-34x+15$.

(3) $8x^2-2x-15$. (4) $12x^2+25x+12$.

(5) $x^2-ax-bx-6a^2+13ab-6b^2$.

(6) $x^2+a^2+b^2-c^2-2ax+2bx-2ab$.

(7) $x^2-3ax+2x+2a^2-5a-3$.

(8) $x^2+2ax-(b+c)x+a^2-ab-ac+bc$.

(9) $x^2+(a+p)x-(b+q)x+(p-q)(a-b)$.

(10) $4x^2-4ax+4bx+a^2+b^2-2ab-c^2$.

(11) $x^2-y^2-z^2+2yz+x-y+z$.

(12) $x^2-y^2+3ax+2yz-ay-z^2+az+2a^2$.

21. For what value of n will the equation $x^2-5x+\frac{1}{2}n=0$ have equal roots? x is a real quantity.

[The algebraical quantities mentioned in the following examples are always to be considered *real*, except when the opposite is mentioned.]

22. For what value of n will the expression $x^2-(n-1)x+n+\frac{1}{4}$ be a perfect square?

23. For what value of a will the equation $x^2-(3a-1)x+2a^2+6a-56=0$ have equal roots?

24. For what values of a will the quadratic expression $x^2-(5a-2)x+4a^2+10a+25$ be a perfect square?

25. The equation $(b^2-ac)x^2-2(a^2-bc)x+(c^2-ab)=0$ have equal roots, find the relation between a, b, c.

26. The quadratic expression $x^2+2(a+b+c)x+3(ab+ac+bc)$ is a perfect square, find the relation between a, b, c.

27. For what values of y will the quadratic expression $x^2-3y(x-1)+4(x-1)+y^2-1$ be a perfect square?

28. If the equations $ax^2+bx+c=0$, $bx^2+cx+a=0$ have a common root, find the relation between a, b, c.

29. If $x-r$ be a common factor of ax^2+bx+c, and $a^2x^2+b^2x+c^2$, shew that $b^2(a-b)(b-c)=ac(a-c)^2$.

30. If the equations $ax^2+bx+c=0$ and $(a+c)(a+b)x^2+(b+c)(b+a)x+(c+b)(c+a)=0$ have a common root, shew that $b^2(c^2-a^2)^2=ac(b^2-c^2)(a^2-b^2)$.

31. If $x-\dfrac{b}{c}$ be the square root of $x^2-\dfrac{2ax}{b}+\dfrac{c}{b}$, find the relation between a, b, c.

32. Given $x^2-ax+b=0$, $x^2-cx+d=0$; one root is common to both and the second equation have equal roots; shew that $2(b+d)=ac$.

33. For what value of a will the equation $x^2-6xy+6x+8y^2-20y+a=0$ have rational roots?

34. For what values of a will the equation $x^2-5xy+ax+4y^2-7y+3=0$ have rational roots?

35. For what values of a will the quadratic expression $x^2-(ay-1)x-6a^2y^2-13ay-a$ have rational simple factors?

36. The equation $x^2-2(yz-b)x+2(a+c)yz-3k+b^2=0$ has rational roots and $k=ab+ac+bc$, shew that $a^2+b^2+c^2=k$.

37. If the quadratic expression $x^2+2bcy^2+c^2z^2-2(b+c)xy-2a^2z^2-2(a^2+b^2)zy$ be resolvable into rational simple factors, shew that $a^4+b^4+c^4=2a^2c^2-b^2c^2$.

38. The quadratic expression $x^2+2xy+2axz-8y^2+5ayz+3z^2$ is resolvable into rational simple factors, find the values of a.

39. If the roots of the equation $bx^2+(a-b-c)x+c=0$ be possible and different, shew that the roots of the equation $(b-c)^2x^2+(a-c)^2+b^2=(2ac-a^2)x^2+2b(2ax+c)$ will be impossible, and vice versâ.

40. For all real values of x shew that in the equation $x^2-2(3y-1)x+8y^2-10y+6=0$, y cannot lie in value between 1 and -5.

41. Shew that for all real values of x in the equation $x^2-4xy+5y^2+2x-12y+16=0$, y must lie between 3 and 5.

42. In the equation $x^2-2(y-2)x+2y^2-5y-2=0$, shew that for all real values of x, y must lie between 3 and -2.

43. In the equation $x^2-2(2y-3)x+3y^2-7y+3=0$, shew that for all real values of x, y cannot lie between 2 and 3.

44. Shew that for all real values of x the expression $\dfrac{x^2-2x+2}{x^2-x+1}$ must lie between 2 and $\frac{2}{3}$.

45. Shew that for all real values of x the expression $\dfrac{2x^2-2x+4}{x^2-4x+3}$ cannot lie between 1 and -7.

46. Shew that for all real values of x the expression $\dfrac{m^2x^2+1}{(m^2+1)x}$ cannot lie between $\dfrac{2m}{m^2+1}$ and $-\dfrac{2m}{m^2+1}$.

47. If $a>b$, for all real values of x, shew that $\dfrac{x^2-2bx+a^2}{x^2+2bx+a^2}$ must lie between $\dfrac{a+b}{a-b}$ and $\dfrac{a-b}{a+b}$.

48. Shew that for all real values of x and y in the equation $x^2+4y^2-4xy-16y+4x+36=0$, y cannot be less than 4 and x cannot be less than 5 respectively.

49. Shew that no other real values of x and y besides 4 will satisfy the equation $x^2-xy+y^2-4x-4y+16=0$.

50. In the equation $x^2+12xy+4y^2+4x+8y+20=0$, shew that for all real values of x and y, y cannot lie between -1 and $\frac{1}{4}$ and x cannot lie between -2 and 1 respectively.

51. In the equation $2x^2-5xy+2y^2+2x+2y-2=0$ shew that for real values of x and y, x and y must not lie between $\frac{10}{3}$ and $\frac{2}{3}$.

52. Find the limits to the real values of x and y which can satisfy the equation $x^2+12xy+4y^2-26x-44y+89=0$.

Problems producing Quadratic Equations.

204. Prob. 1. A person bought a certain number of oxen for 80 guineas, and if he had bought 4 more for the same sum, each of them would have cost a guinea less; required the number of oxen and the price of each.

Let x be the number, then $\frac{80}{x}$ is the price of each, by the question, $\frac{80}{x+4}=\frac{80}{x}-1$.

$$\therefore\ 80=\frac{80x+320}{x}-x-4,$$

$$\therefore\ 80x=80x+320-x^2-4x.$$

$$\therefore\ x^2+4x=320.$$

$$\therefore\ x^2+4x+4=324.$$

$$\therefore\ x+2=\pm 18.$$

$$\therefore\ x=-2\pm 18=16 \text{ or } -20.$$

$$\frac{80}{x}=\frac{80}{16}=5 \text{ guineas, the price of each.}$$

In this, and in many other cases, especially in the solution of philosophical questions, we deduce, from the algebraical process, answers which do not correspond with the conditions. The reason seems to be, that the algebraical expression is more general than the common language, and the equation which is a proper representation of the conditions, will also express other conditions, and answer other suppositions. In the foregoing instance, x may either represent a positive or a negative quantity, and cannot in the operation represent a positive quantity alone (Art. 186); and the equation $\frac{80}{x+4}=\frac{80}{x}-1$, when x is negative, or represents the diminution of the stock, will be a proper expression for the solution of the following problem :—A person sells a certain number of oxen for 80 guineas, and had he sold 4 fewer for the same sum, he would have received a guinea a piece more for them; required the number sold.

205. Prob. 2. To divide a line of 20 inches into two such parts, that the rectangle under the whole and one part, may be equal to the square of the other part.

Let x be the greater part, then will $20-x$ be the less, and $x^2=(20-x)20=400-20x$, by the question.

$$x^2+20x=400.$$

$$x^2+20x+100=400+100=500\ ;$$

$$\therefore\ x+10=\pm\sqrt{(500)}.$$

$$\therefore\ x=\sqrt{(500)}-10 \text{ or} -\sqrt{(500)}-10.$$

The observation contained in the preceding article may be applied here; and it is to be remarked, that the negative values thus deduced are not insignificant, or useless. Here the negative, value shews, that if the line be produced $\sqrt{(500)}+10$ inches, the square of the part produced is equal to the rectangle under the line given, and the line made up of the whole and the part produced.

206. Prob. 3. To find two numbers, whose sum, product and the sum of whose squares, are equal to each other.

Let $x+y$ and $x-y$ be the numbers;
their sum $=2x$; their product $=x^2-y^2$;
the sum of their squares $=2x^2+2y^2$.

By the question $2x=2x^2+2y^2$,

$$\text{or } x=x^2+y^2$$

also $2x=x^2-y^2$.

$$\text{therefore } 3x=2x^2$$

$$\therefore\ x=\tfrac{3}{2}.$$

$$2x=x^2-y^2$$

$$\text{or } 3=\tfrac{9}{4}-y^2.$$

$$\therefore\ y^2=\tfrac{9}{4}-3=\frac{9-12}{4}=\frac{-3}{4}.$$

$$\therefore\ y=\pm\tfrac{1}{2}\sqrt{(-3)}.$$

$$x+y=\tfrac{1}{2}\{3\pm\sqrt{(-3)}\}$$

$$x-y=\tfrac{1}{2}\{3\mp\sqrt{(-3)}\}.$$

Since the square of every quantity is positive, a negative quantity has no square root; the conclusion therefore shews that there are no such numbers as the question supposes.

Prob. 4. A person bought cloth for ~~Rs~~ 675, which he sold at ~~Rs~~ 48 per piece, and gained by the bargain as much as one piece cost him; required the number of pieces.

Let $x=$ number of pieces.

Then $\dfrac{675}{x}=$ cost price of each piece.

$48x-675=$what he gained by the bargain.

By the question, $48x-675=\frac{675}{x}$.

$\therefore 48x^2-675x=675.$

$\therefore x^2-\frac{225}{16}x=\frac{225}{16}.$

$\therefore x=15$ or $-\frac{15}{16}$.

The second value is not applicable.

Prob. 5. A and B set out at the same time to a place at a distance of 150 miles. A travels 3 miles an hour faster than B, and arrives at his journey's end 8 hours and 20 minutes before him. At what rate did each person travel per hour?

Let $x=$rate at which B travels;

then $x+3=$ „ „ A „

$\frac{150}{x}$ and $\frac{150}{x+3}$ are the number of hours for which B and A travel respectively.

But A is 8 hrs. 20 min. or $8\frac{1}{3}$ hours sooner at his journey's end than B.

Hence $\frac{150}{x+3}+8\frac{1}{3}=\frac{150}{x}$.

$\therefore x=6$ or -9.

$\therefore x=6$ miles an hour is the rate of B.

$x+3=9$ „ „ „ „ „ „ A.

The negative value is to be rejected.

Prob. 6. A person sells a horse for ~~Rs~~144 and gains as much per cent. as the horse cost him. What did the horse cost him?

Let x represent in ~~Rs~~ the original cost of the horse.

$x\times\frac{x}{100}$ or $\frac{x^2}{100}=$the gain.

By the question $x+\frac{x^2}{100}=144$.

$\therefore x=80$ or -180.

$\therefore$ ~~Rs~~80$=$the original cost.

Prob. 7. A person bought a certain number of long-cloth pieces for ~~Rs~~120, if he had got 3 pieces more for the

money, he would have paid Rs 2 less for each piece. How many pieces did he buy ?

Let x be the number of pieces he bought.

Then $\frac{120}{x}$ = price paid for each.

$\frac{120}{x+3}$ = supposed price.

By the question $\frac{120}{x+3}=\frac{120}{x}-2$.

$\therefore\ x=12$ or -15.

$\therefore$ 12 is the number of pieces bought.

Prob. 8. A person bought coffee for Rs 40, and sugar for a similar sum, and obtained of the latter 20 seers more than of the former. He sold 40 seers of coffee and 12 seers of sugar, and gained, at the rate of 20 per cent, Rs 4. How many seers of each did he buy ?

Let x = number of seers of coffee bought for Rs 40.

$x+20$ = " " " " sugar " " "

$\frac{40}{x}$ = cost price of each seer of coffee,

$\frac{40}{x+20}$ = " " " " " " sugar,

By the question $\left(40\times\frac{40}{x}+12\times\frac{40}{x+20}\right)\frac{20}{100}=4$.

$\therefore\ x=100$ or -16.

$x+20=120$.

The negative value is to be rejected.

Prob. 9. There is a certain number consisting of three digits ; the digit in the unit's place is double that in the hundred's place ; if the number be divided by the sum of the digits the quotient is 22, but if divided by the sum of the product of the extreme digits and 4, the quotient is also 22. To find the number.

Let x, y and $2x$ be the digits ;

$100x+10y+2x$ is the number.

By the question, $\frac{102x+10y}{3x+y}=22$;

$$\text{also}\quad \frac{102x+10y}{2x^2+4}=22.$$

$$\therefore\ 3x=y \text{ and } 51x+5y=22x^2+44.$$

$$\therefore\ 51x+15x=22x^2+44.$$

$$\therefore\ x^2-3x+2=0.$$

$$\therefore\ x=2 \text{ or } 1,\ y=3x=6 \text{ or } 3 \text{ and } 2x=4 \text{ or } 2.$$

$\therefore$ 264 or 132 is the number.

Prob. 10. The sum of two numbers is 7 and the sum of their cubes is 133. To find the numbers.

Let x and y be the numbers.

$$\text{Then}\qquad x+y=7 \qquad (1)$$

$$x^3+y^3=133 \qquad (2)$$

Divide (2) by (1) $x^2-xy+y^2=19$ (A)

Square (1) $x^2+2xy+y^2=49$ (B)

Subtract (A) from (B), $3xy=30$; $\therefore\ xy=10.$

$$\therefore\ x+\frac{10}{x}=7,$$

$\therefore\ x=2$ or 5 and $y=5$ or 2.

Prob. 11. What are mangoes a hundred when 5 more in a Rupee's worth lowers the price 1 rupee per hundred.

Let x = price of 100 mangoes,

then $\frac{x}{100}$ price of each.

$\frac{x}{100} : 1 :: 1 : \frac{100}{x}$ the number bought for 1 Re.

$\therefore\ \frac{100}{x}+5$ or $\frac{100+5x}{x}$ = number (supposed) for Re. 1.

Then $\frac{100+5x}{x} : 100 :: 1 : \frac{100x}{100+5x}$ the supposed price of 100.

$$\text{By the question } x-\frac{100x}{5x+100}=1.$$

$$\therefore\ x=5 \text{ or } -4.$$

$\therefore$ the price of 100 mangoes = ~~Rs~~ 5.

Prob. 12. The fore-wheel of a carriage makes 16 more revolutions than the hind-wheel in passing over 240 feet : but if the circumference of each wheel be increased by 2 feet, it

will make only 10 revolutions more than the hind-wheel in the same space. Find the circumference of each wheel.

Let x = number of feet in the circumference of the hind-wheel,

y = " " " " " " " fore-wheel.

$$\text{By the question, } \left.\begin{aligned} \frac{240}{y} - \frac{240}{x} &= 16 \\ \frac{240}{y+2} - \frac{240}{x+2} &= 10. \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore\ 15x - 15y &= xy \\ 24(x+2-y-2) &= (x+2)(y+2). \end{aligned}\right\}$$

Hence $y = 6$ and $x = 10$.

Prob. 13. A shop-keeper sold 64 seers of pepper and 100 seers of sugar for Rs40; he sold 4 seers more of sugar for Rs4 than he did pepper for Rs5. What was the price of a seer of each?

Let x = price of pepper (in Rs) per seer.

y = " " sugar (in Rs) " "

$\frac{1}{x}$ = number of seers of pepper sold for 1 Re.

$\frac{1}{y}$ = " " " " sugar " " "

$\frac{4}{y}$ = " " " " " " " 4 Rs

$\frac{4}{y} - 4$ or $\frac{4-4y}{y}$ = " " " pepper " " 5 Rs

$$\therefore \frac{4-4y}{5y} = \frac{1}{x} \text{ and } 64x + 100y = 40.$$

$$\therefore y = \tfrac{1}{5} \text{ and } x = \tfrac{5}{16}.$$

Prob. 14. The product of each of three numbers and the sum of the other two are 36, 50 and 56 respectively. Find the numbers.

Let x, y and z be the three numbers, then

$$x(y+z) = 36 \qquad (A)$$

$$y(x+z) = 50 \qquad (B)$$

$$z(x+y) = 56 \qquad (C)$$

Subtract (A) from (B), $yz - xz = 14$.

From (C), $yz + xz = 56$.

$\therefore\ 2yz=70$ and $2xz=42$. $\quad\therefore\ yz=35$ (1), and $xz=21$ (2)

$xy=36-xz=36-21=15 \qquad (3).$

Multiply (1) and (2), then $xyz^2=35\times21$, and divide it by (3),

then $\qquad z^2=49$; $\therefore\ z=\pm7$.

Hence $y=\pm5$ and $x=\pm3$.

Prob. 15. An officer can form his men into a solid rectangle. He can also form his men into two hollow rectangles of similar length and breadth but 10 deep. If his men be reinforced by 100 men, the whole number can be formed into a solid square of the same perimeter as that of the rectangle; to find the number of men.

Let $x=$ length of the rectangle.

$y=$ breadth " "

$xy=$ number of men.

$\sqrt{(xy+100)}=$ a side of the square.

Also $\frac{1}{4}(2x+2y)=\frac{1}{2}(x+y)=$ a side of the square.

$xy=2\{xy-(x-20)(y-20)\}$,

and $\sqrt{(xy+100)}=\frac{1}{2}(x+y)$.

Hence $x=80$, $y=60$, $\therefore\ xy=4800$.

EXAMPLES LXX.

1. Divide the number 30 into two such parts that their product shall be 200.

2. What two numbers are those whose sum is 30, and product 125?

3. The difference of two numbers is 6; and if 47 be added to twice the square of the less, it will be equal to the square of the greater. What are the numbers?

4. There is a certain number, the product of whose third, fourth and sixth parts is equal to twice the number; required the number.

5. Find two numbers in the proportion 2 : 3, the sum of whose squares is equal to 208.

6. Find two numbers such that their sum shall be 10 and the sum of their squares 58.

7. Find two numbers such that their sum shall be 9 and the sum of their cubes 189.

8. A number is formed by the product of three consecutive even integers, and if it be divided by each of them, the sum of the quotients is 44. Find the number.

9. Find the number the square root of which exceeds the fourth root by 12.

10. Divide 60 into two such parts, that their product shall be to the sum of their squares in the ratio of 2 to 5.

11. There is a rectangular field, whose length exceeds its breadth by 14 yds.; and it contains 3200 sq. yds.; find its dimensions.

12. A person bought a quantity of cloth for Rs 337 8 as., which he sold at Rs 24 per piece, and gained by the bargain as much as one piece cost him; required the number of pieces.

13. A labourer dug two trenches, one 4 yds. longer than the other for Rs 25, and the digging of each cost as many annas per yard as there were yards in its length; find the length of each.

14. A gentleman left Rs 210 to three servants, to be divided in continued proportion, so that the first shall have Rs 90 more than the last; find their legacies.

15. The joint stock of two partners is Rs 1000, one leaves his money in the partnership for 12 months, the other for 18 months; but each takes Rs 990 for capital and profit. What stock did each invest?

16. Two men start from two places, A and B (whose distance is 910 yds.), at a uniform pace, towards each other. If one of them starts 2 minutes 30 seconds before the other, they will meet midway; but if both start at the same time from A and B, after 20 minutes the distance between them will be 370 yds. Find the time in which each man would pass from A to B.

17. A traveller having to walk 45 miles, finds that if he increases his speed $\frac{1}{2}$ a mile an hour, he will perform his task $1\frac{1}{4}$ hours sooner than if he walked at his usual rate. What is that rate?

18. A cistern can be filled with water by two pipes; by one of them it would be filled 6 hours sooner than by the other, and by both together in 4 hours; find the time in which each pipe alone would fill it.

19. A bought certain pieces of cloth for Rs 90, but had he received 6 pieces more for the same money, each piece would have cost him 12 as. less. How many pieces did he buy?

20. A and B distribute ~~Rs~~ 1200 each among a certain number of persons. A relieves 40 persons more than B, and B gives ~~Rs~~ 5 a piece to each person more than A. How many persons were relieved by A and B respectively?

21. A company dining at a hotel find their bill amounts to ~~Rs~~ 87 8 as.; if there had been two more of the party, each of them would have had ~~Rs~~ 5 less to pay than what he had now to pay. Find the number of guests.

22. What are oranges a score when 16 more in a rupee's worth lowers the price 1 anna per score?

23. The forewheel of a carriage makes 30 more revolutions than the hind-wheel in passing over 360 yds.; but if the circumference of each wheel be increased by 1 yd., it will make only 18 revolutions more than the hind-wheel in the same space. Find the circumference of each wheel.

24. A and B engaged to reap equal quantities of wheat, and A began half an hour before B; at 12 o'clock they rested an hour, having finished half the whole work; also B's part was completed at 7 o'clock and A's at a quarter before 10. Determine the times at which they commenced.

25. Two merchants A and B jointly invested £200 in business. A lets his money remain 17 months, and received back in capital and profit £ 171; B allowed his money to remain 12 months and received in capital and profit £ 104. How much did each advance?

26. A and B bought railway shares to the amount altogether of ~~Rs~~4900; they sell out at *par*, A at the end of 4 years and B at the end of 5, and each receives in capital and profit ~~Rs~~3000. How much did each lay out?

27. There is a number consisting of three digits; when the number is divided by the product of the first two digits, the quotient is 28. If 99 times the first digit be added to the number, the number will be inverted; also the sum of three times the second digit and the first is equal to 16. Find the number.

28. A person bought a number of sheep for ~~Rs~~80, and if he had bought four more for the same money, he would have paid Re. 1 less for each. How many did he buy?

29. A regiment of 1000 men is to be drawn up into a rectangular form, so that the difference of the numbers in the two unequal sides may be 117; required the numbers in rank and file.

30. A person bought a certain number of sheep for Rs 90. Having lost 8 of them, and sold the remainder at Rs 4 a-head profit, is no loser by the bargain. How many sheep did he buy?

31. A grazier bought as many sheep as cost him Rs 66, and after reserving 15 out of the number, sold the remainder for Rs 54, and gained Re 1 a-head by them; how many sheep did he buy?

32. A shop-keeper sold 5 mds. of flour and 3 mds. of sugar for Rs 60; he sold 9 seers more of flour for Rs 3 than he did sugar for Rs 7. What was the price of a seer of each?

33. A man left by his will Rs 46800 to be divided equally among his children; two of them die before the division of property is made, and consequently each child receives Rs 1950 more than it was originally entitled to. Required the number of children.

34. The area of the floor of a room is 320 sq. ft., that of one of the sides, 200 sq. ft.; that of the other 160 sq. ft. Find the length, breadth and height of the room.

35. A and B set out at the same time; A from C to go to D, and B from D to go to C; they meet on the road, when it appears that B has travelled 80 miles more than A, and that at the rate he is travelling, he will reach C in 16 days and that A will arrive at D in 25 days. Find the distance of C from D.

36. A cistern can be filled by two pipes running together in 2 hours and 24 minutes. The larger pipe by itself will fill it 2 hours sooner than the smaller. What time will each pipe take separately to fill it?

37. The contents of two cubical vessels are 728 inches, the height of the two together is 14 inches. What are the contents of each vessel?

38. A and B set out from C and D, A starting 3 hours before B. They meet at 20 miles from D, and A reaches D an hour before B reaches C. The next day B starts early, meeting A, who had then gone $\frac{1}{7}$ of his journey back; and, though delayed 3 hours, B reaches D in time to have gone 28 miles further before A reaches C. Required their rates per hour of journeying.

39. Two boys start from the right angle of a triangular field, and run along the sides with velocities in the ratio of 13 : 11. They meet first in the middle of the opposite side, and again 30 yds. from the starting point. Find the length round the field.

40. Three towns A, B, C, lie at the angles of a right-angled triangle, B at the right angle, and the distance AB being the least of the three. A pedestrian finds that the time of his going from A to B, and then from B to C, exceeds the time from A to C direct by $2\frac{2}{3}$ hours. A coach, which left A four hours after him, and travels thrice as fast, overtakes him 8 miles from B in the way to C, and after passing through C to A, and waiting there $6\frac{2}{3}$ hours, it makes the same circuit, and reaches A again at the same time with the pedestrian, who had rested four hours at C. Find the rate at which he walks.

41. At a review of an army, the troops were drawn up in a solid rectangular mass, when there were just $\frac{1}{4}$ as many men in front as there were spectators. Had the depth, however, been increased by 5 men, and the spectators drawn up with the army, the number of men in front would have been 100 fewer than before. Also the army can be formed into two solid squares a side of which will contain 150 men less than the 10th part of the number of men in front of the first rectangle. Find the force of the army.

42. The men of a regiment can be drawn up into a hollow rectangle 5 deep, the hollow part of the rectangle being double the part occupied by the men. The men can also be drawn up into a hollow square 15 deep, the perimeter containing 10 more men than the semi-perimeter of the rectangle. Find the number of men.

43. Find three numbers such that each multiplied by the sum of the other two will produce 56, 72 and 80 respectively.

44. The sum of four numbers is 14; the sum of the products of the first and second and third and fourth is 26; of the first and third, and second and fourth is 23; of the first and fourth, and second and third 22; find the numbers.

45. A certain number of labourers can remove a heap of stones in 8 hours from one place to another; if there were 8 more labourers, and each carried every hour 5lbs. less, the heap would be removed in 7 hours: but if the labourers were 8 fewer in number, and each carried every hour 11 lbs. more, the heap would be removed in 9 hours. How many men are employed, and how much does each carry?

46. There were two square pieces of ground to be planted with trees also in the form of squares. If on the first piece the trees are planted 5 feet apart, and on the second $4\frac{1}{2}$ feet, 11113 will be required. But if on the first the trees are planted $5\frac{1}{2}$ feet apart, and on the second 6 feet apart, 7816 will be wanted. How many feet long is each piece of ground?

47. A traveller having had to go to a certain place at a distance of 15 miles, thought of starting at 3 A. M. Sometime between 2 and 3 A. M., he looked at his watch and mistook the minute-hand for the hour-hand and the hour-hand for the minute-hand; and supposing he had been late by more than an hour started at once, increasing his usual rate of walking by $\frac{3}{4}$ of a mile per hour, and reached his destination 1 hour, $38\frac{2}{11}$ minutes before time. If the time he mistook were correct, he would have been late by $11\frac{9}{11}$ minutes (though walking at the increased rate). Find the usual rate of his walking and the correct time.

48. A fast train starts from Howrah at 8 o'clock; it overtakes a *goods train* which travels 15 miles an hour, and after having gone 15 miles further, overtakes a *slow passenger*, which travels 20 miles an hour. Another fast train which travels at the same rate as the first, starts from the same station at 9 o'clock, overtakes the *goods train*, and after having gone 45 miles further, overtakes the *slow passenger*, and finds it has travelled 120 miles. Find the rate of the fast train and the time when the first *fast* train overtook the *goods*.

Variation.

207. In the investigation of the relation which varying and dependent quantities bear to each other, the conclusions are more readily obtained by expressing only two terms in each proportion, than by retaining the four.

But though, in considering the variation of such quantities, two terms only are expressed, it will be necessary for the learner to keep constantly in mind that four are supposed; and that the operations, by which our conclusions are in this case obtained, are in reality the operations of proportionals.

208. Def. 1. One quantity is said to *vary directly* as another, when the two quantities depend wholly upon each other, and in such a manner, that if one be changed, the other is changed in the same proportion.

Let A and B be mutually dependent upon each other, in such a way, that if A be changed to any other value a, B must be changed to another value b, such that $A : a :: B : b$, then A is said to vary directly as B.

Ex. 1. Suppose a body to move uniformly, at the rate of 3 feet in one second of time; then in the *first* second the body would pass over 3 feet, in *two* seconds 6 feet, in *three* seconds 9 feet, &c. &c., therefore, whilst the time varies through

1, 2, 3, 4, &c. seconds, the space varies through 3, 6, 9, 12, &c. feet, but the numbers, 3, 6, 9, &c. are respectively in the same ratio with the numbers 1, 2, 3, 4, &c. Therefore, when a body moves uniformly, "the space varies directly as the time."

Ex. 2. Suppose a servant is employed for 8 as. a day then he would receive 8 as. for one day's work, 16 as. for 2 days' work, 24 as. for 3 days' work, etc., that is, "the wages varies directly as day."

Ex. 3. If the altitude of a triangle be invariable, the area varies as the base. For if the base be increased or diminished, the area is increased or diminished in the same proportion.

The sign $\propto$ placed between two quantities signifies that they vary as each other; thus, $A \propto B$, is read *A varies as B.*

When it is briefly said that A varies as B, it is always meant A varies directly as B.

209. Def. 2. One quantity is said to *vary inversely* as another, when the former cannot be changed in any manner, but the reciprocal of the latter is changed in the same proportion.

A varies inversely as B, $\left(A \propto \dfrac{1}{B}\right)$, if, when A is changed to a, B be changed to b, in such a manner that

$$A : a :: \frac{1}{B} : \frac{1}{b}; \text{ or } A : a :: b : B.$$

Ex. 1. One man can finish a work in one hour, two men can finish the same work in half the time or half an hour, three men can finish the work in one-third of the time or in one-third of an hour etc. Thus the number of men *varies inversely* as the time.

Ex. 2. If the area of a triangle be given, the base varies inversely as the perpendicular altitude.

Let P and p represent the altitudes, B and b, the bases of two triangles equal in area; then $\dfrac{P \times B}{2} = \dfrac{p \times b}{2}$ (the area of a triangle being the half of the rectangle under the base and perpendicular) or $P \times B = p \times b$; therefore (Art. 141),

$$P : p :: b : B :: \frac{1}{B} : \frac{1}{b}.$$

210. Def. 3. One quantity is said to *vary as two others jointly*, if, when the former is changed in any manner, the product of the other two will be changed in the same proportion.

Thus, A varies as B and C jointly, ($A \propto BC$), when A cannot be changed to a, but the product BC must be changed to bc, such, that $A : a :: BC : bc$.

Ex. The area of a triangle varies as its base and perpendicular altitude jointly. Let, A, B, P, represent the area, base and perpendicular altitude of one triangle respectively; a, b, p, those of another; then $BP = 2A$, and $bp = 2a$; therefore $\frac{A}{a} = \frac{BP}{bp}$ or $A : a :: BP : bp$.

211. Def. 4. One quantity is said to vary *directly* as a second and *inversely* as a third, when the first cannot be changed in any manner, but the second multiplied by *the reciprocal of* the third is changed in the same proportion.

A varies directly as B, and inversely as C, $\left(A \propto \frac{B}{C}\right)$,

when $A : a :: \frac{B}{b} : \frac{C}{c}$; A, B, C, and a, b, c being corresponding values of the three quantities.

Ex. The base of a triangle varies as the area directly and the perpendicular altitude inversely. The notation in the last Article being retained, $\frac{BP}{bp} = \frac{A}{a}$; and multiplying both sides by $\frac{p}{P}$, we have $\frac{B}{b} = \frac{Ap}{aP} = \frac{A}{P} \div \frac{a}{p}$; therefore, $B : b :: \frac{A}{P} : \frac{a}{p}$.

212. *If one quantity vary as a second, and that second as a third, the first varies as the third.*

Let $A : a :: B : b$, and $B : b :: C : c$, then (Art. 142), $A : a :: C : c$. That is, $A \propto C$. In the same manner, if $A \propto B$ and $B \propto \frac{1}{C}$, then $A \propto \frac{1}{C}$.

213. *If two quantities vary respectively as a third, their sum, or difference, or the square root of their product, will vary as the third.*

Let $A \propto C$ and $B \propto C$, then $(A \pm B) \propto C$; also $\sqrt{(AB)} \propto C$.

By the supposition, $A : a :: C : c :: B : b$; therefore $A : a :: B : b$; alternately, $A : B :: a : b$; and by composition or

division, $A \pm B : B :: a \pm b : b$; alt. $A \pm B : a \pm b :: B : b :: C : c$; that is $A \pm B \propto C$.

$$\text{Again } A : a :: C : c,$$
$$\text{and } B : b :: C : c.$$

therefore $AB : ab :: C^2 : c^2$, (Art. 153.)

and $\sqrt{(AB)} : \sqrt{(ab)} :: C : c$. (Art. 154.)

that is $\sqrt{(AB)} \propto C$.

214. *If one quantity vary as another, it will also vary as any multiple, or part, of the other.*

Let $A \propto B$, and m be any quantity, then because $A : a :: B : b$, $A : a :: mB : mb$, or $A : a :: \frac{B}{m} : \frac{b}{m}$ (Art. 150); that is, $A \propto mB$ or $\propto \frac{B}{m}$.

215. *If A vary as B, A is equal to B multiplied by some invariable quantity.*

For $A : a :: mB : mb$; then $A : mB :: a : mb$; if therefore m be so assumed that $A = mB$, then in all cases, $a = mb$.

Cor. If we know any corresponding values of A and B the constant quantity m may be found.

Let a and b be the two values known, then $m = \frac{a}{b}$; and in general $A = \frac{a}{b} \times B$.

Ex. Let $s \propto t^2$, and when $t = 1$ suppose $s = 16$, then $s = 16t^2$.

216. *If one quantity vary as another, any power or root of the former will vary as the same power or root of the latter.*

Let A vary as B, then $A : a :: B : b$, and by Art. 154 $A^n : a^n :: B^n : b^n$; that is, $A^n \propto B^n$, where n is whole or fractional.

217. *If one quantity vary as another, and each of them be multiplied or divided by any quantity, variable or invariable, the products or quotients will vary as each other.*

Let A vary as B, and let T be any other quantity. Then, by the supposition, $A : a :: B : b$; therefore $AT : at :: BT : bt$, and $\frac{A}{T} : \frac{a}{t} :: \frac{B}{T} : \frac{b}{t}$ (Art. 151.)

Cor. If $A \propto B$, dividing both by B, $\frac{A}{B} \propto \frac{B}{B}$, that is $\frac{A}{B}$ is constant.

218. *If one quantity vary as two others jointly, either of the latter varies as the first directly and the other inversely.*

Let $V \propto FT$, then by Art. 217, $F \propto \frac{V}{T}$ or $T \propto \frac{V}{F}$.

219. *If the product of two quantities be invariable, those quantities vary inversely as each other.*

Let $B \times P$ be constant, $B \times P \propto 1$; by division $B \propto \frac{1}{P}$.

220. *If four quantities be always proportionals, and one or two of them be invariable, we may find how the others vary.*

Ex. Let p, q, r, s, be always proportionals, and let p be invariable, then $s \propto qr$. Because $ps = qr$ (Art. 138), $ps \propto qr$; and since p is constant, $s \propto qr$, (Art. 214.)

221. *If one quantity vary as a second, and a third as a fourth, the product of the first and third will vary as the product of the second and fourth.*

Let $A \propto B$ and $C \propto D$, then $AC \propto BD$.

Because $A : a :: B : b$,

and $C : c :: D : d$.

$AC : ac :: BC : bd$ (Art. 153);

that is, $AC \propto BD$.

222. *When the increase or decrease of one quantity depends upon the increase or decrease of two others, and it appears that if either of these latter be invariable, the first varies as the other, when they both vary, the first varies as their product.*

Let $S \propto V$ when T is given,

and $S \propto T$ when V is given.

When neither T nor V is given, $S \propto TV$.

The variation of S depends upon the variations of two quantities T and V; let the changes take place separately, and whilst T is changed to t, let S be changed to S_1; then by the supposition, $S : S_1 :: T : t$; but this value S_1 will again be changed to s, by the variation of V, and in the same proportion that V is changed; that is, $S_1 : s :: V : v$, and by compounding this with the last proportion, $SS_1 : S_1 s :: TV : tv$; or $S : s :: TV : tv$. (Art. 150.)

Ex. 1. The area of a triangle varies as the base when the height is invariable, also the area of a triangle varies as the height when the base is invariable, when both the base and height vary, the area varies as the product of the numbers which represent the base and height.

223. COR. In the same manner, if there be any number of magnitudes P, Q, R, etc., each of which varies as another, V, when the rest are constant; when they are all changed, V varies as their product.

Ex. If ~~Rs~~48 be the cost of digging a trench 360 ft. long, 4 ft. broad, and 3 ft. deep, what would be the cost of digging a trench 400 ft. long, 9 ft. broad, and 6 ft. deep?

The cost $\propto$ the length of the trench,
when the breadth and depth are the same.

The cost $\propto$ breadth,
when the length and depth are the same.

The cost $\propto$ depth,
when the length and breadth are the same.

When length, breadth and depth are different,
the cost $\propto$ length $\times$ breadth $\times$ depth.

$\therefore$ the cost required : ~~Rs~~48 : : $400 \times 9 \times 6 : 360 \times 4 \times 3$.

$$\therefore \text{ the cost required} = \frac{48 \times 400 \times 9 \times 6}{360 \times 4 \times 3} = 240 \text{ Rs}$$

Examples worked out.

Ex. 1. Given that y varies as x, and that $y=6$ when $x=2$, find the value of y when $x=4$.

Here $y=x$ multiplied by some constant quantity, m suppose.

$\therefore y=mx.$

$\therefore 6=m \times 2. \qquad \therefore m=3.$

$\therefore y=3 \times x=3 \times 4=12.$

Ex. 2. Given that y varies as x inversely, and that $y=3$, when $x=2$, find the value of y when $x=3$.

$$\text{Let } y=\frac{m}{x}; \qquad \therefore 3=\frac{m}{2}; m=6$$

$$\therefore y=\tfrac{6}{3}=2.$$

Ex. 3. If $x+y \propto x-y$, prove that $x^2+y^2 \propto xy$.

Let $x+y=m(x-y)$.

$\therefore x^2+y^2+2xy=m^2(x^2+y^2-2xy)$.

$$\therefore\ (m^2-1)(x^2+y^2)=2(m^2+1)xy.$$

$$\therefore\ x^2+y^2=\frac{2(m^2+1)}{m^2-1}xy.$$

$$\therefore\ x^2+y^2 \propto xy.$$

Ex. 4. If y = the sum of two quantities, the first of which varies as x and the second as x^2 and $y=8$, when $x=2$, and $y=15$ when $x=3$; express y in terms of x.

Let the quantities be mx and nx^2, then

$$y=mx+nx^2.$$

$$\therefore\ 8=2m+4n,$$

$$15=3m+9n;$$

$$\therefore\ n=1 \text{ and } m=2.$$

$$\therefore\ y=2x+x^2.$$

Ex. 5. Given that y varies as the sum of three quantities of which the first is constant, the second varies as x and the third varies as x^2, and $y=21$, 36 and 57 when $x=1$, 2 and 3 respectively, find the equation between y and x.

Let the second quantity $=px$ *and the third* $=qx^2$.

$a+px+qx^2$ = the sum of the quantities.

$$\therefore\ y=m(a+px+qx^2).$$

$$\left.\begin{aligned}21&=ma+mp+mq\\36&=ma+2mp+4mq,\\57&=ma+3mp+9mq.\end{aligned}\right\}$$

$$\therefore\ mq=3,\ mp=6 \text{ and } ma=12.$$

$$\therefore\ y=12+6x+3x^2=3(4+2x+x^2).$$

Ex. 6. Three globes of gold whose radii are $1\frac{1}{2}$, 2, $2\frac{1}{2}$ inches are melted and formed into a single globe. Find its radius, given that the volume of a globe varies as the cube of its radius.

Let v = volume, r = radius and m the constant multiplier,

then $v=mr^3$.

$\therefore$ volume of the first $=m(\frac{3}{2})^3=\frac{27}{8}m.$

" " second $=m(2)^3=8m.$

" " third $=m(\frac{5}{2})^3=\frac{125}{8}m.$

$\therefore$ the sum of the volumes $=\frac{27}{8}m+8m+\frac{125}{8}m=27m=3^3m.$

The vol. of the 4th $=mr^3=3^3m.$

$$\therefore\ r^3=3^3.\quad \therefore\ r=3.$$

$\therefore$ the radius of the new globe = 3 inches.

EXAMPLES LXXI.

1. Given that y varies as x, and that $y=4$ when $x=3$, find the value of y when $x=5$.

2. Given that y varies inversely as x, and that $y=8$ when $x=2$, find the value of y when $x=4$.

3. If x varies as y and $x=10$ when $y=9$, find the relation between x and y.

4. Given that y varies as x and z jointly and that $y=24$, when $x=2$, $z=4$, find the value of y when $x=3$, $z=1$.

5. If Rs 20 be the cost of a piece of carpet $17\frac{1}{2}$ ft. long, 8 ft. broad, what must be the cost of a piece of carpet of similar quality 15 ft. long and 14 ft. broad. (Apply Ex. 4).

6. Given that x varies as y directly and z inversely, and $x=a$ when $y=b$ and $z=c$; find the relation between x, y, z.

7. If the cube of x vary as the square of y, and that $x=4$ when $y=3$; find the relation between x and y.

8. If $a \propto bc^2$, and 2, 3 and 4 be the values of a, b, c respectively, express a in terms of b, c.

9. If $x \propto z$ and $y \propto z$, then shew that both $x+y$ and $\sqrt{(xy)} \propto z$.

10. If $y^2-a^2 \propto x+b$, and $x=a$ when $y=b$, find the value of x when $y=3a$.

11. If $a^2 \propto b+c$, and if when $b=0$, $a=2c$; shew that $a^2=4c(b+c)$.

12. If $c^2 \propto a^2-b^2$, and $b=\sqrt{(a^2-x^2)}$ when $ac=x^2$, find the value of b when $c=\frac{3}{5}x$.

13. If $\dfrac{x}{y} \propto x+y$ and $\dfrac{y}{x} \propto x-y$, shew that x^2-y^2 is constant.

14. If $y \propto \dfrac{z}{x^2}$ and $z \propto \dfrac{v}{y^2}$, shew that x^2y^2 is constant.

15. If $2a+3b \propto 4a+5b$, shew that $a \propto b$.

16. If $a \propto b$, prove that $a^2 \pm b^2 \propto ab$
and $a^2b \pm ab^2 \propto (a+b)^3$.

17. If $x+y \propto x-y$, prove that $x^3+y^3 \propto xy(x \pm y)$.

18. If $a+b \propto c$ and $b+c \propto a$, prove that $a+c \propto b$ and $a^2 \propto bc$.

19. If $x=a+b+c$, where a is constant, b varies as y and c varies as y inversely and $x=3$, $5\frac{1}{3}$ and 7 when $y=1$, 2, 3 respectively, find the equation between x and y.

20. Given that y is equal to the sum of three quantities, the first being constant, the second varies as x and the third as x^2, and $y=8$, 12, 20 when $x=3$, 4, 5 respectively; find the value of y when $x=2$, also find the value of x when $y=32$.

21. Given that $x=$ sum of three quantities of which the first is constant, the second varies as ab and the third as ab^{-1}; and that $x=0$ when $a=b=1$ and $x=6$ when $a=b=2$ and $x=1$ when $a=0$. Express x in terms of a, b.

22. Given that y varies as the sum of two quantities one of which varies as x and the other as x^2, and $y=18$ when $x=1$ and $y=60$ when $x=2$. Find the equation between x and y; also find the values of x when $y=6$.

23. Given that y varies as the sum of three quantities of which the first is constant, the second varies as x and the third as x^2; and that $y=21$ when $x=1$, $y=63$, when $x=2$, and $y=129$ when $x=3$. Find the equation between x and y.

24. Given that y varies as the sum of three quantities the first of which is constant, the second varies as x and the third as x^2, and that $y=3a$ when $x=a$, $y=7a$ when $x=2a$, and $y=13a$ when $x=3a$; find the value of y when $x=(n-1)a$.

25. Two globes of gold whose radii are R and r are melted and formed into a new globe. Find the radius of the new globe, given that the volume of a globe $\propto$ as the cube of its radius.

26. A spherical shell has within it a hollow sphere of half its diameter; compare its weight with that of a solid sphere of the same diameter. Given, the volume of a sphere varies as the cube of its radius.

27. Two circular plates each four inches thick, the diameters of which are 15 and 20 inches are melted and formed into a single circular plate an inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

28. The value of diamonds $\propto$ the square of their weight. A diamond of 2 carats is worth five times the value of a ruby of 4 carats and both together are worth £600. Required the value of a diamond, weighing 4 carats.

29. A locomotive engine without a train can go 24 miles an hour, and its speed is diminished by a quantity which varies as the square root of the number of waggons attached. With four waggons its speed is 20 miles an hour. Find the greatest number of waggons which the engine can move.

30. The number of marks lost by a student in a mathematical paper varies directly as the number of days he was absent in a year. He would have received 75 per cent. of the total number of marks had he been absent for ten days only. He barely passed though absented from the lectures for 24 days. Find the minimum per cent. of the number of marks for passing.

Arithmetical Progression.

224. Quantities are said to be in Arithmetical Progression when they increase or decrease by a common difference.

Thus the following series are in Arithmetical Progression :

$$1, 3, 5, 7, 9, \ldots\ldots$$

$$30, 27, 24, 21, 18, \ldots\ldots$$

$$a, a+b, a+2b, a+3b, \ldots\ldots$$

$$a, a-b, a-2b, a-3b \ldots\ldots$$

In the first example the common difference is 2, in the second -3, in the third b, and in the fourth $-b$.

Hence it is manifest, that if a be the first term and b be the common difference, then $a+b$ is the second term, $a+2b$ the third, $a+3b$ the fourth, &c. and $a+(n-1)b$ the n^{th} term.

225. *The sum of a series of quantities in arithmetical progression is found by multiplying the sum of the first and last terms by half the number of terms.*

Let a be the first term, b the common difference, n the number of terms, l the last term, and s the sum of the series;

Then $a+(a+b)+(a+2b)+(a+3b)+\ldots\ldots+l=s$,

then writing the series in the reverse order,

$l+(l-b)+(l-2b)+(l-3b)+\ldots\ldots\ldots+a=s.$

By adding the terms that are vertically opposite,

$(a+l)+(a+l)+(a+l)+(a+l)+\ldots\ldots+(a+l)=2s$;

and since $a+l$ is repeated n times,

therefore $(a+l)n=2s$.

Hence $s=(a+l)\frac{n}{2}$. (1)

Also $l=a+(n-1)b$ (2)

therefore $s=\{a+a+(n-1)b\}\frac{n}{2}$

$$=\{2a+(n-1)b\}\frac{n}{2} \tag{3}$$

226. COR. *In an Arithmetical Progression the sum of any two terms equidistant from each end of the series is equal to the sum of the first and last terms.*

The truth of this is manifest from the preceding demonstration. We may also prove this in the following manner :—

Let a be the first term d the common difference and l the last term.

Then the n^{th} term from the beginning $=a+(n-1)d$, and the n^{th} term from the end $=l-(n-1)d$; the sum of these terms $=a+l=$ the sum of the first and last terms.

227. Any three of the quantities l, a, n, b, being given the fourth may be found from the equation $l=a+(n-1)b$.

Ex. 1. Let 2 be the first term, 3 the common difference and 10 the number of terms, to find the last term.

$$\begin{aligned}\text{The last term} &= a+(n-1)b\\ &=2+(10-1)3\\ &=2+27\\ &=29.\end{aligned}$$

Ex. 2. Let 3 be the first term, 6 the number of terms and 18 the last term, to find the common difference.

$$l=a+(n-1)b.$$
$$18=3+(6-1)b.$$
$$18-3=5b.$$
$$\therefore\ b=3.$$

Ex. 3. Let 4 be the first term, 2 the common difference and 28 the last term, find the number of terms.

$$28=4+(n-1)2,$$
Hence $n=13$.

Ex. 4. Let 30 be the last term, 3 the common difference and 8 be the number of terms, find the first term.

$$30=a+(8-1)3.$$
$$\therefore\ a=30-21=9.$$

228. To insert a given number of Arithmetical Means between two given quantities.

Let m be the number of terms to be inserted between a and b; let d be the common difference; then, since the number of terms = the number of means + the number of extremes, therefore n (or the number of terms in the progression) $= m+2$.

$$b \text{ (the last term)} = a+(n-1)d = a+(m+2-1)d$$
$$= a+(m+1)d.$$
$$\therefore \ d = \frac{b-a}{m+1}.$$

Hence the required m terms $a+d$, $a+2d$, $a+3d$,............... $a+md$ are obtained.

Ex. 5. To insert 5 Arithmetical Means between 3 and 15.

7 is here the total number of terms. Let b be the common difference, then

$$15 = 3+(7-1)b.$$
$$\therefore \ b = 2.$$

The Arithmetical means are

$3+2$, $3+4$, $3+6$, $3+8$ and $3+10$

or 5, 7, 9, 11 and 13.

229. Any three of the quantities s, a, n, b being given, the fourth may be found from the equation

$$s = \{2a+(n-1)b\}\frac{n}{2}.$$

Ex. 6. To find the sum of 14 terms of the series 1, 3, 5, 7, &c.

Here $a=1$, $b=2$, $n=14$, therefore

$$s = \{1\times2+(14-1)2\}\tfrac{14}{2}.$$
$$=(2+26)7$$
$$=196.$$

Ex. 7. Required the sum of 9 terms of the series, 11, 9, 7, 5, &c.

In this case $a=11$, $b=-2$, $n=9$; therefore,

$$s = \{11\times2+(9-1)\times(-2)\}\tfrac{9}{2}$$
$$=(22-16)\times\tfrac{9}{2}$$
$$=27.$$

Ex. 8. If the first term of an Arithmetical Series be 14, and the sum of 8 terms be 28, to find the common difference and the series.

$$\text{Since } \{2a+(n-1)b\}\frac{n}{2}=s.$$

$$\therefore\ 2a+(n-1)b=\frac{2s}{n}.$$

$$\therefore\ (n-1)b=\frac{2s}{n}-2a.$$

$$\therefore\ b=\frac{2s-2an}{n(n-1)}.$$

In the case proposed, $s=28$, $a=14$, $n=8$;

$$\text{therefore, } b=\frac{2\times28-2\times14\times8}{8\times7}$$

$$=\frac{7-28}{7}=-3.$$

Hence, 14, 11, 8, 5, &c. is the required series.

Ex. 9. The sum of an Arithmetical Progression consisting of 10 terms is 145, and the common difference is 3; to find the first term.

$$\text{Since } \{2a+(n-1)b\}\frac{n}{2}=s.$$

$$\therefore\ 2an=2s-(n-1)bn.$$

$$\therefore\ a=\frac{2s-(n-1)nb}{2n}$$

$$=\frac{145\times2-9\times10\times3}{2\times10}$$

$$=\frac{290-270}{20}=1.$$

Ex. 10. The sum of an Arithmetical series is 42, the common difference is -4 and the first term is 18, to find the number of terms.

$$\{2a+(n-1)b\}\frac{n}{2}=s.$$

$$\therefore\ 2an+bn^2-bn=2s.$$

$$\therefore\ bn^2+(2a-b)n=2s.$$

$$\text{Hence } n=\frac{b-2a\pm\sqrt{\{(2a-b)^2+8sb\}}}{2b}$$

$$=\frac{-4-36\pm\sqrt{\{(36+4)^2-42\times4\times8\}}}{-8}$$

$$=\frac{-40\pm16}{-8}=3\text{ or }7.$$

230. In the preceding exercise *two* values of n are found both of which are applicable. For the progression is

$$18, 14, 10, 6, 2, -2, -6, \&c.$$

and the sum of the first three terms is the same as the sum of the first seven terms.

231. In some cases one of the values of n is positive and the other negative. In such cases the negative value is excluded, since the number of terms can only be expressed by a positive integer.

Ex. 11. The first term of an Arithmetical Progression is 7, the common difference is 2 and the sum of the terms is 27; to find the number of terms.

$$2s = 2an + n(n-1)b.$$
$$\therefore\ 54 = 14n + 2n^2 - 2n$$
$$\therefore\ n^2 + 6n = 27.$$

Hence $n = 3$ or -9.

The series is 7, 9, 11.

If we put 6 more terms before 7 we obtain the series

$$-5, -3, -1, 1, 3, 5, 7, 9, 11,$$

the sum of all these terms is also 27.

But 7 is not here the first term, therefore the negative value is to be excluded.

An explanation for the negative value of n may be obtained in the following manner:—Put $-n$ for n in the original equation $s = \{2a + (n-1)b\}\frac{n}{2}$, then

$$s = -\tfrac{1}{2}n\{2a - (n+1)b\}$$
$$= \tfrac{1}{2}n\{(n+1)b - 2a\}$$
$$= \tfrac{1}{2}n\{2(b-a) + (n-1)b\}$$

From this it is manifest that the right-hand member is the sum of an Arithmetical Progression whose first term is $b-a$, and the common difference b.

Hence the first term in the progression for the negative value is -5 or $2-7$.

232. In some cases, however, only one of the values of n is integral and the other fractional.

Ex. 12. Let 13 be the first term, -4 the common difference and 27 the sum of the terms: find the values of n.

$$2 \times 13n + n(n-1)(-4) = 27 \times 2.$$

Hence $n = 3$ or $4\frac{1}{2}$.

The value $4\frac{1}{2}$ suggests that of the two numbers 4 and 5, one will correspond to a sum greater than 27, and the other to a sum less than 27. The sum of 4 terms $= 28$, the sum of 5 terms $= 25$.

233. To find the sum of the first n natural numbers.

The n^{th} number is n.

$$\therefore\ s=\frac{n}{2}(1+n). \qquad \text{Art. 225.}$$

234. To find the n^{th} term and the sum of the first n odd integers.

The series is 1, 3, 5, 7, 9. ....
The n^{th} term $=1+(n-1)2=2n-1$.

$$\therefore\ s=\frac{n}{2}(1+2n-1)$$
$$=\frac{n}{2}\times 2n=n^2.$$

235. To find the n^{th} term and the sum of the first n even integers.

The series is 2, 4, 6, 8. ....
The n^{th} term $=2+(n-1)2=2n$.

$$\therefore\ s=\frac{n}{2}\{2+2n\}=n(n+1).$$

236. To find the sum of the squares of the first n natural numbers.

Let s be the required sum; then

$$s=1^2+2^2+3^2+4^2+\dots\dots\dots\dots+n^2.$$

Because
$$\begin{aligned} n^3-(n-1)^3 &= 3n^2-3n+1,\\ (n-1)^3-(n-2)^3 &= 3(n-1)^2-3(n-1)+1,\\ (n-2)^3-(n-3)^3 &= 3(n-2)^2-3(n-2)+1,\\ \dots\dots\dots &= \dots\dots\dots\\ 3^3-2^3 &= 3\cdot3^2 \quad -3\cdot3 \quad +1,\\ 2^3-1^3 &= 3\cdot2^2 \quad -3\cdot2 \quad +1,\\ 1^3-0^3 &= 3\cdot1^2 \quad -3\cdot1 \quad +1. \end{aligned}$$

By addition

$$n^3=3\{1^2+2^2+3^2+\dots\dots+(n-2)^2+(n-1)^2+n^2\}$$
$$-3\{1+2+3+\dots\dots+(n-2)+(n-1)+n\}+n$$

$$\therefore\ n^3=3\{1^2+2^2+3^2+\dots\dots+n^2\}-3\times(n+1)\frac{n}{2}+n$$

$$=3s-3\frac{n(n+1)}{2}+n$$

$$\therefore\ 3s=n^3+\frac{3n(n+1)}{2}-n=\frac{n(n+1)(2n+1)}{2}.$$

$$\therefore\ s=\tfrac{1}{6}n(n+1)(2n+1).$$

237. To find the sum of the cubes of the first n natural numbers.

Let s be the required sum; then

$$s=1^3+2^3+3^3+\dots\dots\dots+n^3.$$

Because

$$\begin{aligned}
n^4-(n-1)^4 &= 4n^3 - 6n^2 + 4n-1,\\
(n-1)^4-(n-2)^4 &= 4(n-1)^3-6(n-1)^2+4(n-1)-1,\\
(n-2)^4-(n-3)^4 &= 4(n-2)^3-6(n-2)^2+4(n-2)-1,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
3^4-2^4 &= 4\cdot3^3-6\cdot3^2+4\cdot3-1,\\
2^4-1^4 &= 4\cdot2^3-6\cdot2^2+4\cdot2-1,\\
1^4-0^4 &= 4\cdot1^3-6\cdot1^2+4\cdot1-1.
\end{aligned}$$

By addition

$$\begin{aligned}
n^4 &= 4(1^3+2^3+3^3+\dots+n^3)-6(1^2+2^2+3^2+\dots+n^2)\\
&\qquad +4(1+2+3+\dots+n)-n\\
&=4s-6\times\tfrac{1}{6}n(n+1)(2n+1)+4(1+n)\tfrac{1}{2}n-n\\
&=4s-n(n+1)(2n+1)+2n(n+1)-n.
\end{aligned}$$

$$\therefore 4s=n^4+2n^3+n^2$$

$$\therefore\ s=\tfrac{1}{4}n^2(n+1)^2=\{\tfrac{1}{2}n(n+1)\}^2$$

Cor. Because $1+2+3+\dots+n=\frac{1}{2}n(n+1)$;
therefore $1^3+2^3+3^3+\dots+n^3=\{1+2+3+\dots+n\}^2$.

EXAMPLES LXXII.

1. Find the last term of each of the following series.

(1) 3, 5, 7...to 20 terms.

(2) 4, 7, 10...to 32 terms.

(3) $12\frac{1}{4}$, $14\frac{1}{2}$, $16\frac{3}{4}$...to 40 terms.

(4) $\frac{7}{6}$, $\frac{1}{6}$, $-\frac{5}{6}$...to 10 terms.

(5) $-9\frac{1}{2}$, $-6\frac{1}{2}$, $-3\frac{1}{2}$...to 12 terms.

(6) 1, 4, 7...to n terms.

(7) 1, 3, 5.........to $2n$ terms.

(8) 2, 4, 6.........to $2n$ terms.

(9) $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$.........to n terms.

(10) $\frac{1}{3}$, $\frac{7}{6}$, 2.........to $2n-1$ terms.

(11) $(x+y)^2$, x^2+y^2, $(x-y)^2$.........to n terms.

(12) $\frac{a-b}{a+b}$, $\frac{3a-2b}{a+b}$, $\frac{5a-3b}{a+b}$......to n terms.

2. The sum of an Arithmetical Progression of 40 terms is 1600, and the common difference is 2; find the first term.

3. In an Arithmetical series the first term is 28, and the sum of 8 terms is 140; find the common difference.

4. The sum of an Arithmetical series is 1240, the common difference -4 and the number of terms 20; find the first term.

5. The sum of an Arithmetical series is 1455, the first term 5, and the number of terms 30; find the common difference.

6. Insert 6 arithmetical means between 2 and 16.

7. Insert 7 arithmetical means between $\frac{1}{2}$ and -1.

8. Insert 4 arithmetical means between 1 and -1.

9. The 4th term of a series is 29, the 7th is 50; find the first term and the common difference.

10. Sum to 16 terms 2, 5, 8, 11,.........

11. Sum to 12 terms 12, 8, 4.........

12. Sum to 36 terms 40, 38, 36, 34,.........

13. Sum to 20 terms 15, 11, 7, 3.........

14. Sum to 32 terms $1, 1\frac{1}{2}, 2, 2\frac{1}{2}$,.........

15. Sum to 150 terms $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}$, ...

16. Sum to 100 terms 1, 3, 5, 7,.........

17. *Sum to 8 terms* $-7, -4, -1$,.........

18. Sum to 16 terms $\frac{1}{2}, \frac{3}{8}, \frac{1}{4}$,............

19. Sum to $2n$ terms $\frac{1}{3}, \frac{5}{6}, \frac{4}{3}$,............

20. Sum to n terms $1-\frac{1}{n}, 1-\frac{2}{n}, 1-\frac{3}{n}$,.........

21. Sum to n terms $1-n, 1-\frac{n}{2}, 1$,.........

22. Sum to n terms $1-n, 2-\frac{3}{4}n, 3-\frac{1}{2}n$,.........

23. The sum of n terms of an arithmetical series of which 1 is the first term, is $4n^2-3n$; find the common difference.

24. The sum of n terms of an arithmetical series, of which the common difference is 10, is $5n^2-4n$; find the first term.

25. *A* travels uniformly at the rate of 6 miles an hour and sets out on his journey 3 hours, 20 minutes before *B*; *B* follows him at the rate of $3\frac{1}{2}$ miles the first hour: $4\frac{1}{2}$ miles the second, $5\frac{1}{2}$ the third, and so on. In how many hours will *B* overtake *A*?

26. There are n arithmetic means between 3 and 17, and the last is 3 times as great as the first; find the number of means.

27. The sum of a series in Arithmetical Progression is 140, the first term is 28 and the common difference -3; find the number of terms.

28. Given $a=7$, $b=2$, $s=567$; find n.

29. A traveller bound to a place at the distance of 198 miles, goes 30 miles the first day, 28 the second, 26 the third, and so on. In how many days will he arrive at his journey's end?

30. Find three numbers in Arithmetical Progression, such that their sum is 9, and the sum of their cubes 153.

31. The sum of three numbers in Arithmetical Progression is 15, and the sum of the squares of the two extremes is 58. What are the numbers?

32. There are four numbers in Arithmetical Progression, the sum of the squares of the extremes is 272, and the sum of the squares of the means is 208. What are the numbers?

33. There are four numbers in Arithmetical Progression, the sum of the two extremes is 8, and the product of the means 15. What are the numbers?

34. The shortest side of a right-angled triangle is 12 inches, and its three sides are in arithmetical progression; find the other sides.

35. One hundred stones are placed on the ground, in a straight line, at a distance of 2 feet from each other. How far does a person travel who brings them one by one to a basket standing at a distance of 20 yards from the first stone?

36. Between the numbers 3 and 27 as extremes, to find a series of arithmetic means whose sum shall be 75.

37. The sum of the first nine terms of an arithmetic series whose first term is unity, is one-third of the sum of the following nine terms; find the series.

38. The sum of an arithmetical series is 507, the last term is 75, and the common difference 6; find the number of terms.

39. The n^{th} term of an arithmetical series is $\frac{1}{6}(3n-1)$; shew that the sum of n terms $=\frac{1}{12}n(3n+1)$.

40. If S_1, S_2, S_3 be the sums of n terms of three arithmetical series, of each of which 1 is the first term and the common differences are 1, 2, and 3 respectively; prove that $S_1+S_3=2S_2$.

41. The sum of the first two terms of an arithmetical progression is 4, and the fifth term is 9: find the series.

42. Divide 25 into five parts, which are in Arithmetical Progression, and such that the sum of the squares of the least and greatest of them is less by 1 than the sum of the squares of the other three.

43. The first term of an Arithmetical Progression is n^2-n+1, and the common difference is 2; prove that the sum of n terms is n^3.

If the first term be $n^{m-1}-n+1$, the sum of n terms $=n^m$.

44. If a be the first term, and l be the last term, of an Arithmetical Series, and if an arithmetic Mean be inserted between every two successive terms of the series, prove that the sum of the original series exceeds the sum of the arithmetic means by $\frac{1}{2}(a+l)$.

45. The latter half of $2n$ terms of an Arithmetical Series is equal to one-third of the sum of $3n$ terms of the same series.

46. If from a series of n terms in Arithmetical Progression, another series be formed such that each term is an Arithmetic Mean between the consecutive terms of the first series; and if S and S_1 be the sums of these series respectively, prove that $S : S_1 :: n : n-1$.

47. The difference between the sums of m and n terms of an arithmetical progression is to the sum of $m+n$ terms as $m-n$ is to $m+n$.

48. Find the sum of the first n numbers of the form $3m+1$.

49. If $s=2pn+rn^2$ whatever be the value of n, find the m^{th} term and the sum of m terms.

50. If the p^{th} term be P and q^{th} term be Q, find the sum of n terms of P, Q, p, q, n.

51. There are r arithmetical progressions, each beginning with 1, and their common differences are 1, 2, 3 &c. r; shew that the sum of their n^{th} terms $=\frac{1}{2}\{(n-1)r^2+(n+1)r\}$.

52. If s_1, s_2, s_3, &c, be the sums of r arithmetical series each to n terms, the first terms being 1, 2, 3, &c. and the common differences 1, 3, 5, &c.; prove that $s_1+s_2+s_3+$&c. $=\frac{1}{2}rn(rn+1)$.

53. If, in an arithmetical series, the $(m+n)^{\text{th}}$ term $=p$, and $(m-n)^{\text{th}}=q$; then prove that the m^{th} term $=\frac{1}{2}(p+q)$ and the n^{th} term $=p-\frac{1}{2}(p-q)\frac{m}{n}$.

√ 54. There is a number of series in $A.\ P.$ whose common differences are 1, 2, 3, &c.; prove that, if the sum of r terms of each of these be r^2, their first terms will form a decreasing arithmetical series, whose first term is $\frac{1}{2}(r+1)$ and the common difference $-\frac{1}{2}(r-1)$.

55. If S_r denote the sum of r terms of an arithmetical series, then will $\frac{1}{2}n(n-1)\ S_{2n}-n(n-2)S_n-S_{n^2}$.

56. Sum to $2n$ terms $1-3+5-7+\ldots\ldots\ldots$

57. If the sum of r terms of an arithmetical series be to the sum of p terms as $r^2 : p^2$; shew that the r^{th} term will be to the p^{th} term as $2r-1 : 2p-1$.

58. Sum to $2n+1$ terms $1-2+3-4+\ldots\ldots\ldots$

59. $\dfrac{a}{b-c}$, $\dfrac{b}{c-a}$, $\dfrac{c}{a-b}$ are in Arithmetical Prog.; prove that $2(a^3+c^3-2b^3)=(a+b+c)(a^2+c^2-2b^2)$,

60. Sum to n terms $1\times2+2\times3+3\times4+4\times5+\ldots\ldots\ldots$

61. Divide unity into four parts in $A.\ P.$ so that the sum of their cubes shall be equal to $\frac{1}{10}$.

62. Sum to n terms $a^2+(a+b)^2+(a+2b)^2+\&c.$

63. If s_n be generally the sum of n terms of an $A.\ P$, shew that $s_{n+3}-3s_{n+2}+3s_{n+1}-s_n=0$.

Geometrical Progression.

238. Def. Quantities are said to be in *Geometrical Progression*, or continued proportion, when the first is to the second as the second to the third, and as the third to the fourth, &c., that is, when every succeeding term is a certain multiple, or part, of the preceding term. Thus the following series are in Geometrical progression :—

$$1,\ 2,\ 4,\ 8,\ 16,\ldots\ldots\ldots$$
$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ldots\ldots\ldots$$

In the first example the common ratio is 2 and in the second $\frac{1}{2}$.

If a be the first term, and ar the second, the series will be

$a,\ ar,\ ar^2,\ ar^3,\ ar^4$, &c., the n^{th} term being ar^{n-1}.

For $a : ar :: ar : ar^2 :: ar^2 : ar^3$, &c.

$$r=\frac{ar}{a}=\frac{ar^2}{ar}=\frac{ar^3}{ar^2}=\&c.$$

239. The constant multiplier by which any term is derived from the preceding, is called the *common Ratio*, and it may be found by dividing the second term by the first, *or any other term by that which precedes it.*

Ex. 1. 1, 4, 16, 64, &c., are in Geometrical Progression, find the common ratio.

The common ratio $=\frac{4}{1}=\frac{16}{4}=\frac{64}{16}=4$.

Ex. 2. $1\frac{1}{3}, \frac{1}{3}, \frac{1}{12}, \frac{1}{48}$, &c. are in Geometrical Progression, find the common ratio.

The common ratio $= \frac{1}{3} \div 1\frac{1}{3} = \frac{1}{4}$.

240. *If quantities be in geometrical progression, their differences are in geometrical progression.*

Let a, ar, ar^2, ar^3, ar^4, &c. be the quantities; their differences, $ar-a$, ar^2-ar, ar^3-ar^2, ar^4-ar^3, &c., form a geometrical progression, whose first term is $ar-a$, and common ratio r.

241. *Quantities in geometrical progression are proportional to their differences.*

For $a : ar :: ar-a : ar^2-ar :: ar^2-ar : ar^3-ar^2$, &c.

242. *In any geometrical progression, the first term is to the third as the square of the first to the square of the second.*

Let a, ar, ar^2, ar^3, &c. be the progression, then

$$a : ar^2 :: a^2 : a^2r^2.$$

243. In the same manner it may be shewn, that the first term is to the $(n+1)^{\text{th}}$ term, as the first raised to the n^{th} power to the second raised to the same power.

244. *If any terms be taken at equal intervals in a geometrical progression, they will be in geometrical progression.*

Let $a, ar, \ldots ar^n, \ldots ar^{2n}, \ldots ar^{3n}, \ldots$&c. be the progression, then a, ar^n, ar^{2n}, ar^{3n}, &c. are at intervals of n terms, and form a geometrical progression, whose common ratio is r^n.

245. *If the two extremes, and the number of terms in a geometrical progression be given, the means may be found.*

Let a, and b be the extremes, n the number of terms, and r the common ratio; then the progression is a, ar, ar^2, $ar^3 \ldots ar^{n-1}$; and since b is the last term, therefore

$$ar^{n-1} = b \text{ and } r^{n-1} = \frac{b}{a}.$$

Therefore $r = \left(\frac{b}{a}\right)^{\frac{1}{n-1}}$; and r being thus known, the terms of the progression ar, ar^2, ar^3, &c. are known.

Ex. There are four means or intervening terms, in a geometrical progression between 2 and 64; find them.

Here $n=6$, $a=2$ and $l=64$.

$$\therefore r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = 32^{\frac{1}{5}} = 2;$$

and the means are 4, 8, 16 and 32.

246. *To find the sum of a series of quantities in Geometrical Progression.*

Let a be the first term, r the common ratio, n the number of terms, and s the sum of the series.

Then $s = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-2} + ar^{n-1}$;
and $rs = \quad ar + ar^2 + \ldots\ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n$.

Subtracting, $rs - s = ar^n - a$.

$$\therefore s = \frac{ar^n - a}{r-1} = a\frac{r^n - 1}{r-1}.$$

Cor. 1. If l be the last term, $l = ar^{n-1}$,

$$\therefore s = \frac{rl - a}{r-1}.$$

247. Cor. 2. From the equation $s = \dfrac{rl-a}{r-1}$, any three of the quantities, s, r, l, a, being given, the fourth may be found.

248. Cor. 3. When r is a proper fraction, as n increases the value of r^n, or of ar^n, decreases, and when n is increased without limit, ar^n becomes less, with respect to a, than any magnitude that can be assigned. Hence the sum of the series which in general is $\dfrac{ar^n - a}{r-1}$, or $\dfrac{ar^n}{r-1} - \dfrac{a}{r-1}$, is reduced in this case, to $\dfrac{-a}{r-1}$ that is, $\dfrac{a}{1-r}$.

The quantity, $\dfrac{a}{1-r}$, which we call the sum of the series is the limit to which the sum of the terms approaches, but never actually attains; it is however the true representation of the series continued *sine fine*, for this series arises from the division of a by $1-r$; therefore $\dfrac{a}{1-r}$ may without error be substituted for it.

Ex. 1. To find the sum of 20 terms of the series, 1, 2, 4, 8, &c.

Here $a=1$, $r=2$, $n=20$; therefore

$$s = \frac{1 \times 2^{20} - 1}{2-1} = 2^{20} - 1.$$

Ex. 2. Required the sum of 12 terms of the series 64, 16, 4, &c.

Here $a=64$, $r=\frac{1}{4}$, $n=12$; therefore

$$s = \frac{64 \times (\frac{1}{4})^{12} - 64}{\frac{1}{4} - 1} = \frac{64 - 64 \times (\frac{1}{4})^{12}}{1 - \frac{1}{4}} = \frac{64 \times 4^{12} - 64}{4^{12} - 4^{11}}$$

$$= \frac{64}{4^{11}} \times \frac{4^{12} - 1}{4 - 1} = \frac{1}{4^8} \cdot \frac{4^{12} - 1}{3}.$$

Ex. 3. To sum to 12 terms the series $1, -3, 9, -27$, &c.

In this case, $a=1$, $r=-3$, $n=12$,

$$\text{therefore } s=\frac{(-3)^{12}-1}{-3-1}=-\frac{3^{12}-1}{4}.$$

Ex. 4. Find the sum of the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+$ &c. to *infinity*.

$$\text{Here } a=1, r=-\tfrac{1}{2}; \quad \therefore s=\frac{1}{1+\frac{1}{2}}=\tfrac{2}{3}.$$

249. If the common ratio in a geometrical progression continued to infinity be less than unity, each term bears a constant ratio to the sum of all which follow it. Let $a+ar+ar^2+ar^3+\cdots$ be the series, then the n^{th} term is ar^{n-1}; the sum of all the terms which follow ar^{n-1}.

$$=ar^n+ar^{n+1}+ar^{n+2}+ar^{n+3}+\ldots\ldots\ldots$$

$$=ar^n(1+r+r^2+r^3+\ldots)=\frac{ar^n}{1-r}.$$

The ratio of the n^{th} term to the sum of all which follow it is $ar^{n-1}\div\frac{ar^n}{1-r}=\frac{1-r}{r}$. This is constant whatever n may be.

250. Recurring decimals are quantities in geometrical progression, where $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c., are the common ratios according as one, two, three, &c. figures recur; and the vulgar fraction corresponding to such a decimal, is found, by summing the series.

Thus for example $\cdot123123123\cdots$&c. denotes

$$\frac{123}{1000}+\frac{123}{1000^2}+\frac{123}{1000^3}+\ldots\ldots;$$

here the common ratio is $\frac{1}{1000}$.

$$\text{Therefore it is equal to } \frac{\frac{123}{1000}}{1-\frac{1}{1000}}=\frac{123}{999}.$$

Otherwise, let $s=\cdot123123123\ldots\ldots\ldots$

$$\therefore\ 1000s=123\cdot123123123\ldots\ldots\ldots$$

Subtracting, $999s=123$.

$$\therefore\ s=\frac{123}{999}.$$

251. *To find the value of a mixed recurring decimal.*

Let the non-circulating part be represented by a number N consisting of n digits, and a period R consisting of r digits, then the decimal fraction

$$=\frac{N}{10^n}+\frac{R}{10^{n+r}}+\frac{R}{10^{n+2r}}+\frac{R}{10^{n+3r}}+\ldots$$

$$=\frac{N}{10^n}+\frac{R}{10^{n+r}}\left(1+\frac{1}{10^r}+\frac{1}{10^{2r}}+\ldots\right)$$

$$=\frac{N}{10^n}+\frac{R}{10^{n+r}}\times\frac{10^r}{10^r-1}=\frac{(N\times10^r+R)-N}{(10^r-1)10^n}.$$

Ex. To find the value of $\cdot243\dot{4}\dot{6}$

$$\text{Value}=\frac{N\times10^r+R-N}{(10^r-1)10^n}=\frac{243\times10^2+46-243}{(10^2-1)10^3}$$

$$=\frac{24346-243}{99000}=\frac{24103}{99000}.$$

Otherwise :—Let $x=\cdot243464646\ldots\ldots\ldots$

then $100000x=24346\cdot4646\ldots\ldots\ldots$

and $1000x=243\cdot4646\ldots\ldots\ldots$

$\therefore\ 99000x=24346-243=24103.$

$\therefore\ x=\frac{24103}{99000}.$

252. There are some series which are not actually composed of terms in Arithmetical or Geometrical Progression, but whose sum may however be easily obtained by applying the rules laid down in the preceding articles.

Ex. 1. To find the n^{th} term and the sum to n terms of $2+6+14+30+\&c.$

$$n^{\text{th}}\text{ term}=2^{n+1}-2.$$

$$\text{The series}=(2^2-2)+(2^3-2)+(2^4-2)+\cdots+(2^{n+1}-2)$$
$$=2^2+2^3+2^4+\ldots-(2+2+2+2\cdots\text{to } n \text{ terms})$$

$$=\frac{2^2(2^n-1)}{2-1}-2n=2^{n+2}-4-2n.$$

Ex. 2. To find the sum of a series of quantities in geometrical progression, having their coefficients in arithmetical progression.

Let $a+(a+b)r+(a+2b)r+\ldots$ be the series.

Let $s=a+(a+b)r+(a+2b)r^2+\ldots+\{a+(n-1)b\}r^{n-1}$

$\therefore\ sr=ar+(a+b)r^2+\ldots+\{a+(n-2)b\}r^{n-1}+\{a+(n-1)b\}r^n$

By subtr. $s(1-r)=a+br+br^2+\ldots+br^{n-1}-\{a+(n-1)b\}r^n$
$$=a+br(1+r+r^2+\ldots r^{n-2}+r^{n-1})-(a+nb)r^n.$$

$$=a+\frac{br(1-r^n)}{1-r}-\{a+nb\}r^n.$$

$$\therefore\ s=\frac{a-\{a+nb\}r^n}{1-r}+\frac{br(1-r^n)}{(1-r)^2}.$$

Ex. 3. To find the sum of

$$1+2x+3x^2+\dots+(n-1)x^{n-2}+nx^{n-1}.$$

Let $s=1+2x+3x^2+\dots+(n-1)x^{n-2}+nx^{n-1}$

$\therefore\ sx=x+2x^2+\dots+(n-2)x^{n-2}+(n-1)x^{n-1}+nx^n.$

$\therefore\ s(1-x)=1+x+x^2+\dots+x^{n-1}-nx^n.$

$$=\frac{1-x^n}{1-x}-nx^n.$$

$$\therefore\ s=\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}.$$

Ex. 4. To sum to n terms $5+55+555+\dots\dots$

The last term $=5\times10^{n-1}+5\times10^{n-2}+\dots$

$+5\times10^2+5\times10+5\times1$

$=5(10^{n-1}+10^{n-2}+\dots+10^2+10+1)$

$$=5\times\frac{10^n-1}{9}=\frac{5}{9}(10^n-1).$$

Let $s=5+55+555+\dots\dots$

$10s=50+550+\dots\dots+\frac{50}{9}(10^n-1).$

$\therefore\ 9s=\frac{50}{9}(10^n-1)-(5+5+5+5+\dots$ to n terms)

$\therefore\ 9s=\frac{50}{9}(10^n-1)-5n.$

$\therefore\ s=\frac{50}{81}(10^n-1)-\frac{5}{9}n.$

EXAMPLES LXXIII.

1. Sum to 12 terms $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\dots$
2. Sum to 10 terms $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots$
3. Sum to 12 terms $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$
4. Sum to 10 terms $\frac{2}{3}-\frac{1}{3}+\frac{1}{6}-\frac{1}{12}+\dots$
5. Sum to n terms $9+6+4+\dots$
6. Sum to n terms $9+15\frac{3}{4}+27\frac{9}{16}+\dots\dots$
7. Sum to infinity $a+b+\frac{b^2}{a}+\dots$, when $a>b$.
8. Sum to infinity $100+40+16+\dots$
9. Sum to infinity $2-1\frac{1}{2}+1\frac{1}{8}-\dots$
10. Sum to infinity $\frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\dots$

11. Sum to infinity $a-\frac{a^2}{b}+\frac{a^3}{b^2}-\frac{a^4}{b^3}+\ldots$, when $b>a$.

12. Sum to infinity $2+2a+2a^2+2a^3+\ldots$

13. Sum to infinity $\frac{3}{2}-1+\frac{2}{3}-\frac{4}{9}+\cdots$

14. Sum to 8 terms $\sqrt{2}+\sqrt{8}+4\sqrt{2}+\ldots$

15. Sum to 10 terms $\sqrt{2}+\sqrt{3}+\sqrt{(4\frac{1}{2})}+\ldots$

16. Sum to 8 terms $\frac{2}{3}-\sqrt{\frac{2}{3}}+1-\ldots$

17. Sum to 8 terms $\sqrt{\frac{3}{5}}-\sqrt{6}+2\sqrt{(15)}-\ldots$

18. Sum to n terms $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}+\frac{2}{3}\sqrt{2}+\ldots$

19. Sum to n terms $\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2\sqrt{2}}+\ldots$

20. Sum to infinity $\frac{a}{x}\sqrt{\frac{3}{2}}+\sqrt{\frac{a}{x}}+\sqrt{\frac{2}{3}}+$&c.

21. Sum to infinity $\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{1}{2-\sqrt{2}}+\frac{1}{2}+\cdots$

22. Sum to infinity $\frac{1}{3}+\frac{2}{3^2}+\frac{1}{3^3}+\frac{2}{3^4}+\cdots$

23. Sum to n terms $1+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\ldots\ldots$

24. Sum to n terms $1+\frac{3}{3}+\frac{5}{3^2}+\frac{7}{3^3}+\frac{9}{3^4}+\ldots$

25. Sum to n terms $2x+\frac{3}{2}x^2+\frac{4}{4}x^3+\frac{5}{8}x^4+\ldots$

26. Sum to n terms $1-\frac{2}{2}+\frac{3}{4}-\frac{4}{8}+\ldots\ldots\ldots$

27. Sum to infinity $x+\frac{2}{3}x^2+\frac{3}{3^2}x^3+\frac{4}{3^3}x^4+\frac{5}{3^4}x^5+\ldots$

28. The fifth term of a geometrical series is 8 times the second, and the sum of the first and third is 15, find the terms.

29. There are three numbers in geometrical progression whose product is 8 and the sum of their cubes is 73. Find the numbers.

30. Find three numbers in geometrical progression, such, that their sum shall be 7 and the sum of their squares 21.

31. The sum of four numbers in geometrical progression is 30, and the last term divided by the sum of the middle terms is $\frac{4}{5}$; find the numbers.

32. If a, b, c be in A. P., and x, y, be the geometric means between a, b, and b, c, respectively, prove that b^2 is an arithmetic mean between x^2 and y^2.

33. Insert four geometrical means between $\frac{1}{9}$ and 27.

34. The limit of an infinite series is $\frac{3}{5}$, and the sum of the first two terms $\frac{14}{25}$. What is the series?

35. The population of a country increases annually in *G.P.*, and in four years was raised from 10000 to 14641 souls, by what part of itself was it annually increased?

36. The limit of an infinite geometrical series is 2, and the second term is $-\frac{3}{2}$; find the series.

37. The sum of ~~Rs~~ 700 was divided among four persons, whose shares were in geometrical progression; and the difference between the greatest and the least was to the difference between the means as 37 is to 12. What were the respective shares.

38. If the arithmetic mean between a and b be twice as great as the geometric mean; prove that $(2-\sqrt{3})a=(2+\sqrt{3})b$.

39. If a, b, c, d, &c. be n quantities in geometrical progression then will $\frac{1}{a^2-b^2}, \frac{1}{b^2-c^2}, \frac{1}{c^2-d^2}$, &c. be in geometrical progression.

40. A certain number consists of three digits in geometrical progression. The sum of the digits is 13, and if 792 be added to the number, the digits will be inverted. Find the number.

41. Required the sum of the first p terms of a series, whose n^{th} term is $na+a^n$.

42. The second and fourth terms of a series in *G. P.* are $\frac{21}{32}$ and $\frac{189}{512}$; find the limit of the sum of an infinite number of terms of each of the two series satisfying these conditions.

43 Shew that $a-ar+ar^2-ar^3+$......to infinity is to $a+ar+ar^2+ar^3+$......to infinity as $1-r$ is to $1+r$.

44 If $S=1+R+R^2+$ &c. to infinity and

$s=1+r+r^2+$ &c. to infinity;

find the sum of $1+Rr+R^2r^2+$ &c. to infinity in terms of S and s, R and r being less than unity.

45. If $S=1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+$ &c., to infinity and

$s=1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+$ &c., to infinity;

shew that $S:s::27:1$.

46. If a, b and c be the m^{th}, n^{th} and r^{th} terms of a geometrical progression; then prove that $a^{n-r}b^{r-m}c^{m-n}=1$.

47. Sum to n terms $\left(a-\frac{1}{a}\right)^2+\left(a^2-\frac{1}{a^2}\right)^2+\left(a^3-\frac{1}{a^3}\right)^2+\ldots$

48. If the terms of an arithmetic progression $a, a+b, a+2b,$......to infinity be multiplied by the corresponding terms

of a geometric progression, $c, cr, cr^2, \ldots\ldots$ to infinity, find the sum of the resulting series, r being less than unity.

49. If S, the sum of the first $2n$ terms of a geometrical series, whose first term is a and the common ratio r, be equal to x times the $(n+1)^{\text{th}}$ term; shew that $x = \dfrac{Sr^n}{a+(r-1)S}$.

50. If S and S_1 be respectively the sum of n terms of the series $a+ar+ar^2+\ldots\ldots$, and $a+ar^{-1}+ar^{-2}+\ldots\ldots$ and if l be the last term of the first series, then shew that $aS = lS_1$.

51. If P be the product, S the sum, and s the sum of the reciprocals of n quantities in geometrical progression, then shew that $P^2 = \left(\dfrac{S}{s}\right)^n$.

52. If $s_1, s_2, s_3 \ldots\ldots s_n$ be the sums of n geometrical progressions whose first terms are $a, 2a, 3a \ldots\ldots na$ respectively and the common ratio in each progression is r, find the sum of $s_1+s_2+\ldots\ldots\ldots+s_n$.

53. If $s_1 = 1 + \dfrac{1}{r} + \dfrac{1}{r^2} + \ldots\ldots\ldots$ to infinity,

$$s_2 = 1 - \frac{1}{r} + \frac{1}{r^2} - \ldots\ldots\ldots\ldots\ldots\ldots$$

$$s_3 = 1 + \frac{1}{r^2} + \frac{1}{r^4} + \ldots\ldots\ldots\ldots\ldots$$

shew that $s_1 \times s_2 = s_3$.

Harmonical Progression.

253. Def. Any magnitudes a, b, c, d, e; &c. are said to be in Harmonical Progression, if $a : c :: a-b : b-c$; $b : d :: b-c : c-d$; $c : e :: c-d : d-e$; &c.

254. *The reciprocals of quantities in Harmonical Progression are in Arithmetical Progression.*

Let a, b, c, &c. be in Harmonical Progression,

then by def. $a : c :: a-b : b-c$;

$\therefore ab - ac = ac - bc$, (Art. 138)

and dividing both sides by abc.

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}.$$

Again $b : d :: b-c : c-d$;

$$\therefore\ bc-bd=bd-cd,$$

and dividing by bcd, $\frac{1}{d}-\frac{1}{c}=\frac{1}{c}-\frac{1}{b}$;

and $\frac{1}{c}-\frac{1}{b}$ has been proved equal to $\frac{1}{b}-\frac{1}{a}$; therefore the quantities $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$, have a common difference; that is, they are in Arithmetical Progression. And the same proof may be extended to any number of terms.

255. *To insert a given number of harmonic means between two given terms.*

Let a and l be two terms and n the number of harmonic means to be inserted. Since the reciprocals of harmonical terms are in arithmetical progression, therefore the problem is reduced to the following :—to insert n arithmetical means between $\frac{1}{a}$ and $\frac{1}{l}$; the reciprocals of the arithmetic means are the harmonic means.

Let b be the common difference.

$$\frac{1}{l}=\frac{1}{a}+(n+1)b.$$

$$b=\left(\frac{1}{l}-\frac{1}{a}\right)\div(n+1)$$

$$=\frac{a-l}{al(n+1)}.$$

$\therefore\ \frac{1}{a}+\frac{a-l}{al(n+1)},\ \frac{1}{a}+\frac{2(a-l)}{al(n+1)}$, etc. are the arithmetic means;

$\therefore\ \frac{al(n+1)}{l(n+1)+a-l},\ \frac{al(n+1)}{l(n+1)+2(a-l)}$, etc. are the harmonic means.

256. If A, G, and H respectively be the arithmetic, geometric and harmonic means between any two quantities, to find the relation between A, G, H.

Let x and y be the two quantities.

$$A-x=y-A;\quad \therefore\ A=\tfrac{1}{2}(x+y).$$

$$x : G :: G : y;\quad \therefore\ G=\sqrt{(xy)}.$$

$$x : y :: x - H : H - y ; \quad \therefore H = \frac{2xy}{x+y}.$$

Hence $G^2 = AH$.

$\therefore A : G :: G : H$.

COR. Hence G lies in value between A and H, also $A > G$ and $G > H$.

$$\text{For } A - H = \frac{x+y}{2} - \frac{2xy}{x+y} = \frac{(x+y)^2 - 4xy}{2(x+y)}$$

$$= \frac{(x-y)^2}{2(x+y)} \text{ a positive quantity.}$$

257. There is no method by which the sum of an harmonical series can be found, because the sum of any number of terms cannot be found from the sum of their reciprocals.

EXAMPLES LXXIV.

1. Find the fourth and fifth terms of the series $4 + 2\frac{2}{3} + 2$.

2. Find 16 harmonic means between $\frac{1}{2}$ and $\frac{1}{10}$.

3. Find a harmonic mean between -3 and 15.

4. Find the fourth term of the harmonic series $1\frac{1}{2}, 2, 3$.

5. The harmonic mean is $\frac{8}{9}$ of the arithmetic mean, and one of the numbers is 3; find the other.

6. The difference of the arithmetic and harmonic means between two numbers is 2; and one of the numbers is three times the other; find them.

7. The sum of three terms of an harmonic series is 11, and the sum of their squares 49, find the terms.

8. An arithmetic mean between two numbers is to the geometric mean :: 5 : 3; and the difference between the geometric and harmonic means is $2\frac{2}{5}$; find the numbers.

9. The sum of three terms of an harmonic series is 11 and their product is 36; find the terms.

10. The arithmetic mean between two numbers exceeds the geometric by 5, and the geometric exceeds the harmonic by 4, find the numbers.

11. There are four numbers, the first three are in *A. P.* and the last three in *H P.* shew that the product of the extremes is equal to the product of the means.

12 If a, b, c be the m^{th}, n^{th} and p^{th} terms of an harmonic series, shew that $(m-n)ab + (p-m)ac + (n-p)bc = 0$.

13. The nth term of an H. P. is N and the rth term R, find the $(n+r)$th term.

14. If n harmonic means be inserted between 1 and r, shew that the first : the last : : $n+r : nr+1$.

15. If a, b, c be in harmonic progression, shew that

$$\frac{1}{a}+\frac{1}{c}=\frac{1}{b-a}+\frac{1}{b-c}.$$

16. There are three numbers in harmonic progression, if unity be subtracted from the first, the progression becomes geometric; and if 4 be subtracted from the third, it becomes arithmetic. Find the numbers.

17. If $a^m=b^n=c^r=$&c., and $a, b, c,$ &c. be in geometric progression; then will $m, n, r,$ &c. be in harmonic progression.

18. Between two numbers a and b, n harmonic means are inserted of which c is the first term, shew that $a : b : : (a-c)\,n : c-b$.

19. If $x, y, z,$ be in $A. P.$, $\alpha, \beta, \gamma,$ in $H. P.$ $x\alpha, y\beta, z\gamma$ in $G. P.$, prove that $x : y : z : : \frac{1}{\gamma} : \frac{1}{\beta} : \frac{1}{\alpha}$.

20. If the arithmetic mean between two quantities a and b be n times the harmonic mean; then will

$$a : b : : \sqrt{n}+\sqrt{(n-1)} : \sqrt{n}-\sqrt{(n-1)}.$$

21. If v, x, y, z be four positive quantities in $H. P.$; shew that $v+z>x+y$.

22. If a, b, c be in $H. P.$, shew that

$$\frac{1}{c-b}+\frac{1}{b-a}+\frac{4}{a-c}=\frac{1}{a}-\frac{1}{c}.$$

23. If y^2 be an arithmetic mean between x^2 and z^2, prove that $y+z, x+z, x+y$ are in Harmonic Progression.

24. If b be a geometric mean between a and c, shew that $a+b, 2b$ and $c+b$ are in Harmonic Progression.

EXAMPLES LXXV.

Miscellaneous Examples on the foregoing articles.

1. If $\sqrt{(ax)}+\sqrt{(by)}+\sqrt{(cz)}=0$, prove that $a^2x^2+b^2y^2+c^2z^2 =2(abxy+acxz+bcyz)$.

2. Solve the equations

(1) $x^{\frac{7}{3}}=56x^{-\frac{2}{3}}+x^{\frac{5}{6}}$.

(2) $xyz=48, \frac{x}{yz}=\frac{1}{12}, \frac{xy}{z}=\frac{4}{3}$.

3. Resolve into elementary factors :—

$a^2+b^2+c^2-2ab+2bc-2ac$, and shew that

$(x+y+z)^3-(x^3+y^3+z^3)=3(x+y)(x+z)(y+z)$.

4. Eliminate y and z from the equations

$$e^x+e^y+e^z=p,$$
$$e^{x+y}+e^{x+z}+e^{y+z}=q,$$
$$e^{x+y+z}=r.$$

5. Solve the equations

(1) $\dfrac{x^4+1}{(x+1)^4}=\dfrac{13}{25}$.

(2) $\dfrac{1}{\sqrt{(a-x)}+\sqrt{a}}+\dfrac{1}{\sqrt{(a+x)}-\sqrt{a}}=\dfrac{\sqrt{a}}{x}$.

(3) $x^3-10x^2+31x-30=0$.

6. Sum to n terms

$a+(a-b)r+(a-2b)r^2+(a-3b)r^3+\cdots$

7. If a be the first term, r the common ratio, n the number of terms, s the sum, s_2 the sum of the squares of the terms of a G. P., shew that $s_2(r+1)-s^2(r-1)=2as$.

8. Solve the equations :—

(1) $2\sqrt{\{x(x-4)\}}=1-4x$.

(2) $\left.\begin{aligned}(3x+y)^2-(3y+x)^2&=24\\ x^2+y^2&=5\end{aligned}\right\}$.

(3) $\left.\begin{aligned}4x^2-6xy+y^2&=11\\ 3y^2-2xy&=14\end{aligned}\right\}$

9. Skilled workmen and labourers are employed on a work, a skilled workman receives 1s. 6d. a day more than a labourer. The average of their daily wages exceeds by $1\frac{1}{2}d.$ the sum which would be the average if skilled workmen and labourers were employed in equal numbers. If 6 men of each kind are discharged, the average of the daily wages is raised by $1\frac{1}{2}d.$ Find the number of men of each kind.

10. If a, b, c be the roots of the equation $x^3-7x+6=0$, find the value of

$$a^2\left(\frac{1}{b}+\frac{1}{c}\right)+b^2\left(\frac{1}{a}+\frac{1}{c}\right)+c^2\left(\frac{1}{a}+\frac{1}{b}\right).$$

11. Solve the equations.

(1) $(x-3)(x-2)+(x-3)(x-1)+(x-2)(x-1)=2$

(2) $(ax+by)^2+(ay-bx)^2=2\left\{\frac{a}{b}+\frac{b}{a}\right\}^2$.

$\frac{x}{y}+\frac{y}{x}=2\frac{a^2+b^2}{a^2-b^2}$.

12. Find the factor which will rationalise $\sqrt{a}+\sqrt[3]{b}$.

13. Find the square root of $9-2\sqrt{3}-2\sqrt{5}+2\sqrt{(15)}$.

14. Prove that $\frac{x^2y^2}{(z^2-x^2)(z^2-y^2)}+\frac{x^2z^2}{(y^2-x^2)(y^2-z^2)}$

$$+\frac{y^2z^2}{(x^2-y^2)(x^2-z^2)}=1.$$

15. Solve the equations :—

(1) $x^4-2x^3-3x^2+2x+2=0$, one root being $1+\sqrt{3}$.

(2) $2x^3+3x^2=1$.

(3) $x^3-3x^2+4=0$.

16. Sum to 10 terms $1{\cdot}01+(1{\cdot}01)^2+(1{\cdot}01)^3+\ldots\ldots$

17. If $\frac{x}{y}+\frac{y}{x}=a$, $\frac{y}{z}+\frac{z}{y}=b$, $\frac{z}{x}+\frac{x}{z}=c$; eliminate x, y, z.

18. $a=\frac{-1+\sqrt{(-3)}}{2}$, $b=\frac{-1-\sqrt{(-3)}}{2}$, prove that $a^2+ab+b^2=0$.

19. Solve the equations.

(1) $x^4-2x^3+2x^2-2x+1=0$.

(2) $x^4-x^3-3x^2+5x-2=0$.

20. In the election to an office the numbers of the supporters of the three candidates, A, B, C, are in arithmetical progression. C, who has the fewest supporters, withdraws before the day of election and of his supporters 9 decline to vote, and the rest so divide themselves that 4 times as many of them vote for A as for B. On the whole, 125 votes are given for A and 82 votes for B. Find the number of supporters of each candidate.

21. Simplify $\frac{\{a+b\sqrt{(-1)}\}^3-\{a-b\sqrt{(-1)}\}^3}{\{a+b\sqrt{(-1)}\}^2-\{a-b\sqrt{(-1)}\}^2}$.

22. Sum to n terms

a, $(a+2b)r$, $(a+4b)r^2$, $(a+6b)r^3$, &c.

23. The sum of 5 or 15 terms of an A. P. is 75; find the tenth term.

24. If $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$ and $\frac{x^2}{m^2}+\frac{y^2}{n^2}+\frac{z^2}{p^2}=1$; prove that,

$$\frac{a^2}{m^2}+\frac{b^2}{n^2}+\frac{c^2}{p^2}=\frac{a^2+b^2+c^2}{x^2+y^2+z^2}.$$

25. Eliminate y and z from the following equations

$$\left.\begin{array}{r} x+y+z=p \\ xy+xz+yz=q \\ xyz=r \end{array}\right\}$$

26. Shew that

$\frac{a+b\sqrt{(-1)}}{\{a-b\sqrt{(-1)}\}^2}+\frac{a-b\sqrt{(-1)}}{\{a+b\sqrt{(-1)}\}^2}$ is a possible quantity.

27. If a, b, c, d be in G. P.; prove that

$(a-b+c-d)^2=(a-b)^2+(c-d)^2+2(b-c)^2$.

28. Find the harmonic mean between a and $\frac{ab}{2a-b}$.

29. $\frac{a+b}{c}$, $\frac{a+c}{b}$, $\frac{b+c}{a}$ are in A. P., prove that a, b, c are in H. P.

30. If $x+x^{-1}=2a$, $x-x^{-1}=2b\sqrt{(-1)}$, $y+y^{-1}=2c$ and $y-y^{-1}=2d\sqrt{(-1)}$; find $xy+(xy)^{-1}$, and shew that $(ac-bd)^2+(bc+ad)^2=1$.

31. Find what value of x will make $x^2+2ax+b^2$ the square of $x+c$. What does the result become when $a=b=c$?

32. Prove that $\sqrt[n]{a\sqrt[n]{a\sqrt[n]{a}}}$ to infinity$=a^{\frac{1}{n-1}}$; and that $(1+x+x^2+\ldots$to n terms$)(1-x+x^2-x^3+\ldots$to n terms$)$ $=\frac{1-x^{2n}}{1-x^2}$, when n is an odd integer.

Permutations and Combinations.

258 *Def.* The different orders in which any things can be arranged, are called their *permutations.* *

Thus the permutations of a, b, c taken two and two together, are ab, ba, ac, ca, bc, cb.

259. *Def.* The combinations of things are the different collections that can be formed out of them, without regarding the order in which they are placed.

* Some writers call them permutations in the case when all the things are taken together; in all other cases they call them variations or arrangements.

Thus, ab, ac, bc, are the combinations of the letters a, b, c, taken two and two; ab and ba, though different permutations, forming the same combination.

260. *The number of permutations that can be formed out of* n *things, taken two and two together, is* $n(n-1)$, *taken three and three together, is* $n(n-1)(n-2)$.

In n things, a, b, c, d, &c. a may be placed before each of the rest, and thus form $n-1$ permutations in which a stands first; in the same manner, there are $n-1$ permutations in which b stands first, and so of the rest; therefore there are, on the whole, $n(n-1)$ permutations of this kind ab, ba, ac, ca. &c.

Again, of $n-1$ things b, c, d, &c. taken two and two together, there are $(n-1)(n-2)$ *permutations*, by the former part of the article, and by prefixing a to each of these, there are $(n-1)(n-2)$ *permutations*, taken three and three, in which a stands first; in the same manner of $n-1$ things a, c, d, &c. taken two and two together there are $(n-1)(n-2)$ permutations, and by prefixing b to each of these, there are $(n-1)(n-2)$ permutations, taken three and three together, in which b stands first; therefore there are, on the whole $n(n-1)(n-2)$ such permutations.

Ex. The number of permutations of 6 things taken three together $=6\times5\times4=120$.

261. *To find the number of permutations of* n *things taken* r *at a time.*

Let ${}^nP_r$ represent generally the permutations of n letters, a, b, c, d, &c. taken r at a time.

By Art. 260, ${}^nP_2=n(n-1)$

" " " ${}^nP_3=n(n-1)(n-2)$

Likewise ${}^nP_4=n(n-1)(n-2)(n-3)$

From these cases, we may suppose the law to hold generally and that

$$\begin{aligned}{}^nP_r&=n(n-1)(n-2)\dots\dots\{n-(r-1)\}\\&=n(n-1)(n-2)\dots\dots(n-r+1).\end{aligned}$$

Suppose ${}^nP_{r-1}=n(n-1)(n-2)\dots\dots(n-r+2)$.

$\therefore$ omitting a letter as a and putting $n-1$ for n,

$${}^{n-1}P_{r-1}=(n-1)(n-2)(n-3)\dots\dots(n-r+1).$$

Now, prefixing a before each of these *permutations*, there are $(n-1)(n-2)\dots\dots\dots(n-r+1)$ permutations of n letters where a stands first, so if b had been omitted at first, we may make the same number of permutations where b stands first; so on for the rest of the n letters and

$$\therefore {}^nP_r=n(n-1)(n-2)\dots\dots(n-r+1)$$

If, therefore, the formula be true for ${}^nP_{r-1}$, it is true for ${}^nP_r$: but we have seen that it is true for three at a time, therefore it is true for four at a time, therefore it is true for five at a time, and so on; thus it holds universally.

262. Cor. The number of permutations of n things taken all together is $n(n-1)(n-2)$..................$(n-n+1)=n(n-1)(n-2)$..................$4{\cdot}3{\cdot}2{\cdot}1$; it is generally represented by the symbol $\lfloor n$; $\lfloor n$ is read factorial n.

263. *To find the number of permutations of* n *things taken all together when two of the things are alike, when three of the things are alike, and when* r *of them are alike.*

By Art. 261. ${}^nP_n=n(n-1)(n-2)$.........$4{\cdot}3{\cdot}2{\cdot}1$.

Now of the n things a, b, c, d, etc., let a and b be alike. The permutations of a and b are ab, ba, when a and b are different; but they become aa, that is, one only when a and b are the same. Therefore the total number of permutations of n things taken all together, must be equal to $\lfloor n$ divided by 2 when two of the things are alike.

Again because a, b, c may be arranged in $3\times2\times1$ ways when they are different, but in one way only when a, b, c are alike, *i. e.* aaa. Therefore $\lfloor n$ must be divided by $3\times2\times1$ or by $\lfloor 3$ when three of the things are alike.

Likewise, $\lfloor n$ must be divided by $\lfloor r$ when r of them are alike.

264. *To find the number of permutations of* n *things taken all together which are not all different.*

Let there be n letters and let r of them be a, s of them be b, and t of them be c and the rest unlike.

${}^nP_n$ when r of the n things are alike and the rest different $=\dfrac{\lfloor n}{\lfloor r}$.

But if in addition to the r like things s things of the remaining $n-r$ things be alike, then the total number of permutations will be $=\dfrac{\lfloor n}{\lfloor r}\div\lfloor s$ or $\dfrac{\lfloor n}{\lfloor r\lfloor s}$.

Similarly, when of the n things r, s, t are alike and the rest unlike, the total number of permutations.

$$=\frac{\lfloor n}{\lfloor r\lfloor s\lfloor t}.$$

265. *To find the number of combinations of* n *things taken* t *at a time.*

Let ${}^nC_r$ represent the number of combinations of n things taken r at a time.

${}^nP_2 = n(n-1)$; but each combination of two things admits of two permutations ; therefore there are twice as many permutations as combinations ; therefore

$${}^nC_2 = {}^nP_2 \div 2 = \frac{n(n-1)}{2}.$$

Again, ${}^nP_3 = n(n-1)(n-2)$; but each combination of three things admits of 3.2.1 permutations ; therefore there are 3.2.1 times as many permutations as combinations ; therefore

$${}^nC_3 = {}^nP_3 \div (3.2.1) = \frac{n(n-1)(n-2)}{3.2.1}.$$

Similarly the number of combinations of n things taken r at a time, is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots r}.$$

Multiply both numerator and denominator by $\underline{|n-r}$.

$$\therefore\ {}^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

266. The formula for ${}^nC_r$ may also be obtained in a different manner.

Let N denote the number of combinations of n things taken r at a time, but each combination of r things admits of $\underline{|r}$ permutations. Therefore $N \times \underline{|r}$ represent the whole number of permutations of n things taken r at a time. Wherefore $N \times \underline{|r} = n(n-1)(n-2)\ldots(n-r+1)$.

$$\therefore\ N = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}.$$

267. *The number of combinations of* n *things taken* r *at a time is the same as the number of combinations of* n *things taken* n − r *at a time.*

$${}^nC_{n-r} = \frac{n(n-1)\ldots\ldots\{n-(n-r)+1\}}{\underline{|n-r}}$$

$$= \frac{n(n-1)\ldots\ldots(r+1)}{\underline{|n-r}} \times \frac{\underline{|r}}{\underline{|r}}$$

$$= \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}.$$

But ${}^nC_r = \dfrac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}$; Art. 265.

$$\therefore\ {}^nC_{n-r} = {}^nC_r.$$

268. *To find when the number of combinations of* n *things taken* r *at a time is the greatest.*

${}^nC_r = {}^nC_{r-1}\left(\frac{n-r+1}{r}\right)$; the number will increase so long as $n-r+1 \not< r$; the number will be at its maximum when $n-r+1 = r$ or $r+1$. If $n-r+1=r$, $r=\frac{1}{2}(n+1)$; then n is odd, because r must be an integer, and ${}^nC_r = {}^nC_{n-(n-r+1)} = {}^nC_{r-1}$.

Therefore there are two combinations both greater than any other, one when $r=\frac{1}{2}(n+1)$, and the other when n things are taken $r-1$ or $\frac{1}{2}(n-1)$ at a time.

When n is even, r will be obtained by making $n-r+1 = r+1$, or $r=\frac{1}{2}n$; therefore, when n is even, the greatest number of combinations is when $\frac{n}{2}$ things are taken together.

269. *To find the number of combinations of* n *sets of things in which there are* p *of one sort,* q *of another,* r *of a third, &c., one being taken out of each set, every time.*

First, let there be two sets of things, p of one sort, and q of another. It is evident that if one thing of the former set be taken and combined with each one of the second set, q combinations will be formed. Likewise, if every one of the former set be taken and combined with each one of the second set, altogether pq combinations will be formed. Let there be another set of r things, combining one of this set with the pq combinations of the other two sets, we get, pq combinations. Likewise combining each of the third set with the pq combinations of the first two sets, we get, on the whole pqr combinations of three sets of things, one being taken out of each set every time. Thus, if there be a fourth set of s things, the total number of combinations $= pqrs$.

270. Cor. The number of permutations of 4 sets of things in which there are p of one sort, q of another, r of a third, s of a fourth, one thing being taken from each set $= pqrs\underline{|4}$.

Examples worked out.

Ex. 1. To find the number of permutations of 5 things a, b, c, d, e taken all together.

The number $= 5.4\,3.2.1 = 120$.

Ex. 2. To find the number of permutations which may be made of the 6 letters of the word *number.*

The number of permutations $= 6.5.4.3\,2.1 = 720$.

Ex. 3. How many words may be made with the letters of the expression $a^3b^4c^2$.

Here we have 9 letters, of which 3 are a's, 4 b's and 2 c's.

$$\therefore \text{ the number of words} = \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\times4\cdot3\cdot2\cdot1\times2\cdot1} = 1260.$$

Ex. 4. To find the number of permutations of the letters in the word *algebra*, taken all together.

7 is the number of letters.

If all the letters were different the number of permutations $= 7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$;

but a occurs twice;

$$\therefore \text{ the number} = \underline{|7} \div \underline{|2} = 7\cdot6\cdot5\cdot4\cdot3 = 2520.$$

Ex. 5. The number of permutations of n things taken 3 together is to the number taken 4 together as 1 is to 6; to find n.

$$n(n-1)(n-2) : n(n-1)(n-2)(n-3) :: 1 : 6;$$

$$\therefore 1 : n-3 :: 1 : 6;$$

$$n-3=6.$$

$$n=9.$$

Ex. 6. How many different throws may be made with two dice?

Each of the dice may be thrown in 6 different positions and therefore corresponding to one position of the one, the other may have 6 different positions. Therefore the number of throws $= 6\times6 = 6^2 = 36$.

Ex. 7. In how many ways may 5 men be selected out of 8.

$$\text{The number} = \frac{8\cdot7\cdot6\cdot5\cdot4}{1\cdot2\cdot3\cdot4\cdot5} = \frac{8\cdot7\cdot6}{1\cdot2\cdot3} = 56.$$

Ex. 8. How many different words may be formed out of the letters of the words "*quadratic*" taken 3 at a time.

Excluding one a the number of words that may be formed out of the remaining 8 letters taken 3 at a time $= 8\cdot7\cdot6 = 336$.

The two a's may be combined with each of the remaining 7 letters and thus form 7 collections consisting of 3 letters in each. But each such collection of 3 letters contains 2 a's; therefore each collection may be arranged in $\underline{|3} \div \underline{|2}$ ways, that is, in 3 ways. Therefore the number of words that may be formed out of the 7 collections $= 7\times3 = 21$.

$$\therefore \text{ the total number of words} = 336+21 = 357.$$

Ex. 9. To find the number of words, consisting of two consonants and a vowel, that may be formed out of 19 consonants and 5 vowels.

The combinations of the 19 consonants taken 2 at a time $=\frac{19\times18}{1\times2}=9\times19=171$.

But each combination may be combined with each of the 5 vowels; therefore the total number of collections $=5\times171$.

But each collection consists of 3 letters, therefore each may be arranged in $3\cdot2\cdot1$ ways.

$$\therefore \text{ the number of words}=5\times171\times|\underline{3}=5130.$$

Ex. 10. At a game of cards, 3 being dealt to each person, the number of collections which any one can have is 425 times the number of cards in the pack. To find the number of cards.

Let n be the number of cards,

$$\text{then } \frac{n(n-1)(n-2)}{1\cdot2\cdot3}=425n.$$

$$\therefore n^2-3n=2548.$$

$$\text{Hence } n=52$$

Ex. 11. Into how many different triangles may a hexagon be divided by joining the angular points? Also find the number of diagonals.

The number of triangles $=\frac{6\times5\times4}{1\times2\times3}=20$.

The number of lines joining any two angular points (including the sides of the hexagon) $=\frac{6\times5}{1\times2}=15$.

$\therefore$ the number of diagonals $=15-6=9$.

Ex. 12. Three flags are required to make a signal. How many signals can be given by 24 flags of 6 different colors, there being 4 of each sort?

The signals may be made so that

(1) All the flags will be of the same color;

(2) All the flags will be of different color;

(3) Two are of one color and the remaining one of any other color.

(1) Make a signal of three flags of the same color from the 6 different colors; 6 signals can be so made.

(2) Take 1 flag of each color; the number of ways in which they can be arranged taking 3 at a time $=6\times5\times4=120$.

(3) If 2 flags be taken from any color and 1 flag from each of the remaining colors, then we will get 5 collections of 3 flags in each collection, 2 being of the same color. $\therefore$ if we take 2 flags of each color and 1 from the remaining colors, we will altogether have 5×6 or 30 collections, 2 flags in each collection being of the same color. But each collection of 3 things in which 2 are the same admits of $\frac{3\cdot2}{1\cdot2}$ or 3 different permutations.

$\therefore$ the number of ways in which the signals may be arranged when two of the flags are of one color $=30\times3=90$.

$\therefore$ total number of signals $=6+120+90=216$.

Ex. 13. How many words, each consisting of two vowels and two consonants, can be made out of the letters of the word '*devastation*'? In how many of them will the two *t*'s be together?

The consonants are d, v, s, t, t, n.

The vowels are e, a, a, i, o,

Take the consonants d, v, s, t, n.

and the vowels e, a, i, o.

The number of collns of 5 consonants taken 2 at a time $=10$.

" " " " " 4 vowels " " " " " $=6$.

Now each of the 10 consonant collections may be combined with each of the 6 vowel collections. Therefore the total number of such collections of 4 letters (2 consonants and 2 vowels) $=10\times6=60$. But each collection of 4 letters of these 60 collections may be arranged in $\underline{|4}$ ways. Therefore the total number of permutations $=60\times\underline{|4}=1440$.

But of the consonants the two *t*'s may be taken once and this one collection may be combined with the 6 vowel collections, and each of the 6 such combined collections of 4 letters in which the *t*'s occur twice may be arranged in $\frac{\underline{|4}}{\underline{|2}}$ ways. Therefore the total number of such permutations $=6\times\frac{1}{2}\underline{|4}=3\times4\times3\times2=72$.

Likewise of the vowels the two *a*'s may be taken once and combined with the 10 consonant collections and each of the resulting 10 collections of 4 letters, in which the *a*'s occur twice, may be arranged in $\frac{1}{2}\underline{|4}$ ways. Therefore the total number of such permutations $=10\times\frac{1}{2}\underline{|4}=120$.

Also the two t's and two a's may be arranged in $\frac{\lfloor 4}{\lfloor 2 \lfloor 2}$ ways, that is, in 6 ways.

Therefore the total number of words $=1440+72+120+6$ $=1638$.

Again the two t's may be taken once and when they are to be *together* they may be supposed as one letter and it may be combined with each of the 6 vowel collections, two together, of a, e, i, o. Now each of these 6 collections, containing 3 letters, may be arranged in $\lfloor 3$ ways. $\therefore$ the number of such words $=6\times\lfloor 3=36$.

Also the two a's may be taken once and may be combined with the two t's considered as one t. Therefore the number of such words $=\frac{\lfloor 3}{1\cdot 2}=3$.

$\therefore$ the number of words in which the two t's will be together $=36+3=39$.

EXAMPLES LXXVI.

1. How many different words may be formed of the letters of the word *fraction*?

2. How many different words may be formed of the letters of the words.—*Ganges*, *Calcutta*, *Allahabad*, *Murshidabad*?

3. The number of permutations of n things 2 together is to the number taken 3 together as $1:5$; find n.

4. The number of permutations of n things taken 4 together is 6 times the number taken 2 together; find n.

5. How many changes may be rung with 3 bells out of 10?

6. In how many different ways may 8 persons seat themselves at a round table?

7. The number of permutations of $2n$ things, taken 3 together, is 12 times the number of permutations of n things, 3 together; find n.

8. How many different throws can be made with three dice?

9. How may different words may be formed of the letters of the expression $a^2b^4c^2$; in how many words will a^2 stand first?

10. How many different words may be formed out of the letters in the word '*Algebra*' taken 3 at a time?

11. How often may a different party of 6 be formed from 10 persons?

12. The number of permutations of $2n$ things taken 4 at a time is 252 times the number of permutations of n things taken 2 at a time. Find n.

13. If mangoes are 10 for a Rupee, how many different selections may be made in buying 3 Rupee's worth out of a basket consisting of 32 mangoes? In how many of these will a particular mangoe be found?

14. There are 4 companies of soldiers, in each of which there are 12 men; in how many ways may 4 men be chosen, one being selected out of each company?

15. The number of combinations of n things taken 4 together is to the number taken 2 together as 11 is to 1. Find n.

16. How many words may be formed, consisting of 3 consonants and 2 vowels, out of 10 consonants and 5 vowels?

17. How many different sums may be formed with a guinea, a half-guinea, a crown, a half-crown, a shilling and a sixpence?

18. How many different words consisting of 5 letters in each will be formed out of the 10 letters $a, b, c,$ &c.? In how many will a particular letter occur?

19. The number of combinations of n things taken $m+r$ together is equal to the number of combinations of n things taken $m-r$ together; find n.

20. Into how many different triangles may a decagon be divided by joining the angular points?

21. How many different words consisting of 6 letters in each (4 consonants and 2 vowels) will be formed out of 10 consonants and 4 vowels.

22. The number of combinations of $n+1$ things taken 4 together is $7\frac{1}{2}$ times the number of combinations of $n-1$ things taken 2 together; find n.

23. There are four sets of different things, one containing 4, another 6, the third 8 and the fourth 10; how many different collections can be formed of them, taken four together, one out of each set?

24. The number of combinations of n things 3 together is 24 times the number of combinations of $\frac{n}{2}$ things 4 together. Find n.

25. The number of combinations of n things 3 together is 20 times the number of combinations of m things 4 together; also n is equal to the number of combinations of m things 2 two together. Find m and n.

26. The number of combinations of $4n$ things taken $2n$ together is to the number of combinations of $2n$ things taken n together as $1\cdot3\cdot5\ldots(4n-1)$ is to $\{1\cdot3\cdot5\ldots(2n-1)\}^2$.

27. A cricket club has 20 members; how many different elevens can be formed, and in how many of these will (1) any one member, (2) any three members appear?

28. Out of seven white and eight black sailors, five are to be selected for a boat's crew, which must always consist of three white, and two black men; in how many ways may the crew be formed?

29. In what numbers should 20 men be combined, so as to form the greatest possible numbers of different companies? In how many of these will the same pair of men be found together?

30. Find the number of different triangles into which a polygon of m sides may be divided by joining the angular points; also find the number of diagonals.

31. From a company of 20 soldiers, 4 are selected every night to guard; on how many different nights can a different guard be formed. On how many nights will a particular soldier be engaged?

32. In the above example find the number of nights (1) if a particular soldier be always included to form the guard, (2) if a particular soldier be always excluded.

33. If 64 soldiers are drawn up in a line 8 deep, in how many ways can they be arranged so as to have a different set in the front rank each time? In how many ways if the first rank is always to include the 3 tallest?

34. Find the number of words that can be formed out of 8 letters taken all together, each word being such that three given letters are never separated.

35. There are 24 flags of 4 different colors, 6 being of each color; find the number of signals that may be made by 3 flags at a time.

36. There are twenty flags of 4 different colors, 5 being of each color. How many signals may be made if 4 flags are required to make a signal.

37. There are 30 flags of 5 different colors, 6 being of each color. If 4 flags be required to make a signal, how many signals may be made in each of which two of the flags will be of the same color?

38. A man has 4 coats, 7 waistcoats and 5 pairs of trousers. In how many different suits may he appear.

39. There are 10 gentlemen and 8 servants. How many different parties can be made, each party consisting of three gentlemen and three servants.

40. A gentleman invites 20 friends to dinner, and places 8 at one table and 12 at another, the tables being round. Find the number of ways in which he can arrange the guests.

41. Find the greatest possible number of combinations that can be formed of the 26 letters of the alphabet. In how many of these will all the first 8 letters appear ?

√ 42. How many different sides can be formed in a croquet party consisting of 5 ladies and 3 gentlemen, the gentlemen never being all on the same side.

43. There are 10 guests at a dinner party. The master and mistress of the house have fixed seats opposite one another, and there are two particular guests who must not be placed next to one another, find the number of ways in which the company can be seated.

BINOMIAL THEOREM. Positive Integral Exponent.

271. The method of raising a binomial to any power, by repeated multiplication, has before been laid down. (Art. 82) The same thing may be shown much more expeditiously by the following general rule, which is called the *Binomial Theorem.* The theorem was discovered by Sir Isaac Newton.

Let $x+a$ be the given binomial; and its nth power is

$$x^n + nax^{n-1} + \frac{n(n-1)}{2}a^2x^{n-2} + \frac{n(n-1)(n-2)}{2\cdot 3}a^3x^{n-3} + \&c.,$$

where the index of x, beginning from n is diminished by unity, and the index of a, beginning from 0, is increased by unity, in every succeeding term. Also, the coefficient of each term is found by multiplying the coefficient of the preceding term by the index in that term, and dividing by the index of a increased by unity.

$$\text{Thus } (x+a)^6 = x^6 + 6ax^5 + \frac{6\times 5}{2}a^2x^4 + \frac{6\cdot 5\cdot 4}{2\cdot 3}a^3x^3$$

$$+ \frac{6\cdot 5\cdot 4\cdot 3}{2\cdot 3\cdot 4}a^4x^2 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 3\cdot 4\cdot 5}a^5x + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 3\cdot 4\cdot 5\cdot 6}a^6$$

$$= x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6.$$

272. To investigate this theorem, suppose two, three and four of the quantities $x+a$, $x+b$, $x+c$, &c. be actually multiplied together :—

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc,$$

$$(x+a)(x+b)(x+c)(x+d)=x^4+(a+b+c+d)x^3$$
$$+(ab+ac+bc+ad+bd+cd)x^2$$
$$+(abc+acd+bcd+abd)x+abcd;$$

it is manifest,

(1) That the index of the first term of the product is the same as the number of binomial factors, and in the succeeding terms the index of each term is less than that of the preceding term by unity;

(2) That the coefficient of the first term is unity; the coefficient of the second term is the sum of the second terms of the binomial factors; the coefficient of the third is the sum of the products of every two of the second terms of the binomial factors; the coefficient of the fourth is the sum of the products of every three of the second terms of the binomial factors; and so on; the last term is the product of all the second terms of the binomial factors.

We shall now prove that these laws hold for n binomial factors :—

Suppose these laws to hold for $n-1$ binomial factors; so that,

$$(x+b)(x+c)(x+d)\ldots(x+m)=x^{n-1}+Ax^{n-2}+Bx^{n-3}+\ldots+M,$$

where $A=b+c+d+\ldots$

$B=bc+bd+cd+\ldots$

$C=bcd+bce+bde+\ldots$

$\ldots\ldots=\ldots\ldots\ldots \quad \ldots\ldots$

$M=bcde\ldots m$

Multiply both sides of this identity by a new factor $x+a$; then

$$(x+a)(x+b)(x+c)\cdots(x+m)=x^n+(A+a)x^{n-1}+(B+aA)x^{n-2}$$
$$+(C+aB)x^{n-3}+\ldots+Ma.$$

Therefore $A+a=a+b+c+d\ldots$

= the sum of all the second terms a, b, c, d, &c.

$B+aA=ab+ac+ad+\ldots+bc+bd+cd+\ldots$

= the sum of the products of every two of the terms a, b, c, d, ... m;

$$C+aB=C+a(bc+bd+cd+\ldots)$$
= the sum of the products of every three of the terms a, b, c, d,...m.

$$Ma=abcd\ldots\ldots m.$$
= the product of all the terms a, b, c,......m.

Therefore, if these laws hold when $n-1$ binomial factors are multiplied together, they hold when n factors are multiplied together. But these laws hold when 4 binomial factors are multiplied together, therefore they hold when 5 factors are multiplied together; and if they hold for 5 factors they hold for 6 factors; and so on; therefore they hold for any number of factors.

The product of n binomial factors may be written thus :—

$$(x+a)(x+b)(x+c)\ldots(x+m)$$
$$=x^n+Px^{n-1}+Qx^{n-2}+Rx^{n-3}+\ldots+S.$$

Where $P=a+b+c+\ldots$
$Q=ab+ac+bc+\ldots$
$R=abc+abd+acd\ldots$
$S=abc\cdots m.$

Now, let $a=b=c=$ etc.

then $P=a+a+a+$ &c. to n terms $=na$

$Q=a^2+a^2+a^2+\ldots$ to as many terms as is equal to the number of combinations of n things taken two at a time

$$=\frac{n(n-1)}{2}a^2.$$

$R=a^3+a^3+a^3+\ldots$ to as many terms as is equal to the number of combinations of n things taken three at a time

$$=\frac{n(n-1)(n-2)}{1\cdot2\cdot3}a^3.$$

$S=a.a.a.a\ldots$ to n terms $=a^n$.

And $(x+a)(x+b)(x+c)\ldots(x+m)=(x+a)^n$.

$$\therefore\ (x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1\times2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1\times2\times3}a^3x^{n-3}$$
$$+\ldots\ldots+a^n.$$

The right-hand member of the identity is called the *expansion* of $(x+a)^n$.

273. This theorem may be demonstrated differently. By actual multiplication the successive powers of $x+a$ are found to be as follow :—

$$(x+a)^1=x+a.$$
$$(x+a)^2=x^2+2ax+a^2,$$
$$(x+a)^3=x^3+3ax^2+3a^2x+a^3,$$
$$(x+a)^4=x^4+4ax^3+6a^2x^2+4a^3x+a^4.$$

We assume that the theorem is true for the nth power of $x+a$, then $(x+a)^n$

$$=x^n+nax^{n-1}+\frac{n(n-1)}{1\cdot2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}a^3x^{n-3}+\ldots\ldots$$

Multiply both sides of the identity by $x+a$, then $(x+a)^{n+1}$

$$=x^{n+1}+nax^n+\frac{n(n-1)}{1\cdot2}a^2x^{n-1}+\ldots$$
$$+ax^n+na^2x^{n-1}+\frac{n(n-1)}{1\cdot2}a^3x^{n-2}+\ldots$$
$$=x^{n+1}+(n+1)ax^n+\frac{(n+1)n}{1\cdot2}a^2x^{n-1}+\frac{(n+1)n(n-1)}{1\cdot2\cdot3}a^3x^{n-2}+\ldots$$

Hence the expansion for $(x+a)^{n+1}$ is of the same form as that for $(x+a)^n$, Therefore, if the theorem be true for any exponent it is also true when that exponent is increased by unity. But it is true when the exponent is 4, therefore it is true when the exponent is 5; therefore it is true when the exponent is 6; and so on. Thus the theorem is true for any positive integral exponent.

Ex. 1. To find the expansion of $(x+a)^6$.

Here $n=6$.

$$\therefore \frac{n(n-1)}{1\cdot2}=\frac{6\times5}{1\cdot2}=15, \frac{n(n-1)(n-2)}{1\cdot2\cdot3}=\frac{6\cdot5\cdot4}{1\cdot2\cdot3}=20,$$
$$\frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}=\frac{6\cdot5\cdot4\cdot3}{1\cdot2\cdot3\cdot4}=15,$$
$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot2\cdot3\cdot4\cdot5}=\frac{6\cdot5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4\cdot5}=6;$$
$$\therefore (x+a)^6=x^6+6x^5a+15x^4a^2+20a^3x^3+15a^4x^2+6a^5x+a^6.$$

Ex. 2. To expand $(a^2+xy)^5$.

$$(a^2+xy)^5=(a^2)^5+5(a^2)^4xy+10(a^2)^3(xy)^2+10(a^2)^2(xy)^3$$
$$+5(a^2)^1(xy)^4+(xy)^5$$
$$=a^{10}+5a^8xy+10a^6x^2y^2+10a^4x^3y^3+5a^2x^4y^4+x^5y^5.$$

Ex. 3. To expand $(a^2+2x^2)^6$

$$(a^2+2x^2)^6=(a^2)^6+6(a^2)^5(2x^2)+15(a^2)^4(2x^2)^2+30(a^2)^3(2x^2)^3$$
$$+15(a^2)^2(2x^2)^4+6(a^2)^1(2x^2)^5+(2x^2)^6$$
$$=a^{12}+12a^{10}x^2+60a^8x^4+240a^6x^6$$
$$+240a^4x^8+192a^2x^{10}+64x^{12}.$$

274. In the expansion of $(x+a)^n$ put $x=1$ and $a=x$, then

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\ldots\ldots+x^n.$$

Hence the coefficient of the second term is n; the coefficient of the third term is $\frac{n(n-1)}{1\cdot 2}$; and generally the coefficient of the $(r+1)$th term is the number of combinations of n things taken r together, that is,

$$=\frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}$$

$$=\frac{n(n-1)(n-2)\ldots(n-r+1)\times\underline{|n-r}}{\underline{|r}\,\underline{|n-r}}=\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

275. The number of terms in the expansion of $(1+x)^n$ is $n+1$.

The coefficient of the $(r+1)$th term is derived by multiplying the coefficient of the rth term by $\frac{n-r+1}{r}$; hence the rth term will be the last if $n-r+1$ is equal to 0, $\therefore r=n+1$. There is no term after the rth term. Therefore the number of terms is $n+1$, that is, the number of terms is greater by one than the index.

276. To find the $(r+1)$th or general term of the expansion of $(x+a)^n$.

The 1*st* term is x^n,

" 2*nd* term " na^1x^{n-1},

" 3*rd* term " $\frac{n(n-1)}{1\cdot 2}a^2x^{n-2}$

" 4*th* term " $\frac{n(n-1)(n-2)}{2\times 3}a^3x^{n-3}$.

The *index* of x decreases by one in each term; therefore the *index* of x in the $(r+1)$th term is $n-r$.

The index of a increases by one in each term; therefore the index of a in the $(r+1)^{\text{th}}$ term is r.

The coefficient of any term is formed of the factors n, $\frac{n-1}{2}$, $\frac{n-2}{3}$, &c. the number of these being one less than the number which expresses the position of the term;

$\therefore$ the coefficient of the $(r+1)^{\text{th}}$ term will be

$$\frac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)}{\lfloor r}$$

$$\therefore \text{ the } (r+1)^{\text{th}} \text{ term is } \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots\ldots r}x^{n-r}a^r.$$

OBS. The index of a is the last factor of the denominator, and the sum of the indices of x and a, $=n-r+r=n$.

Cor. 1. If $r=n$, then the $(r+1)^{\text{th}}$ term or the $(n+1)^{\text{th}}$ term will be the last term; for it will be equal to

$$\frac{n(n-1)(n-2)\ldots\ldots3.2.1}{1\cdot2\cdot3\ldots\ldots\ldots n}x^0a^n=a^n.$$

Cor. 2. By multiplying both the numerator and denominator of the coefficient of the $(r+1)^{\text{th}}$ term by $\lfloor n-r$ it may be written thus, $\frac{\lfloor n}{\lfloor r\lfloor n-r}$. If $r=n$, this expression takes the form $\frac{\lfloor n}{\lfloor n\lfloor 0}$; but we have found in Cor. 1, that the coefficient is 1; therefore unless we consider $\lfloor 0$ equivalent to 1, the general expression $\frac{\lfloor n}{\lfloor n\lfloor n-r}$ does not hold for the last term.

277. In the expansion of $(x+a)^n$, the coefficients of any two terms equidistant from the beginning and end are the same.

The coefficient of the $(r+1)^{\text{th}}$ term from the beginning is equal to the coefficient of the $(r+1)^{\text{th}}$ term from the end.

Because the number of terms is $n+1$, the $(r+1,)^{\text{th}}$ term from the end, having r terms after it, is the $\{(n+1)-r\}^{\text{th}}$ or $(n-r+1)^{\text{th}}$ term from the beginning; and its coefficient (by Art. 274), putting $n-r$ for r.

$$=\frac{n(n-1)(n-2)\ldots\{n-(n-r)+1\}}{1.2.3.4\ldots\ldots(n-r)}$$

$$= \frac{n(n-1)(n-2)\cdots(r+1)}{1.2.3\ldots\ldots(n-r)} \times \frac{r(r-1)\ldots\ldots 3.2.1}{r(r-1)\ldots\ldots 3.2.1}$$

$$= \frac{\lfloor n}{\lfloor n-r \,\lfloor r}$$

= coefficient of the $(r+1)^{th}$ term from the beginning. (See Art. 276).

278. To find the greatest coefficient in the expansion of $(1+x)^n$.

Let the $(r+1)^{th}$ term have the greatest coefficient.

The coefficient of the $(r+1)^{th}$ term is = the number of combinations of n things taken r at a time. But this number is the greatest when $r = \frac{n}{2}$ if n is even and $r = \frac{1}{2}(n \pm 1)$, when n is odd. (See Art 268).

279. To find the greatest term in the expansion of $(x+a)^n$.

The r^{th} term $= \dfrac{n(n-1)(n-2)\ldots(n-r+2)}{1 \cdot 2 \cdot 3.4\cdots(r-1)} x^{n-r+1}a^{r-1}$.

The $(r+1)^{th}$ term $= \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3.4\cdots(r-1)r} x^{n-r}a^{r}$.

Hence the $(r+1)^{th}$ term is obtained by multiplying the r^{th} term by $\dfrac{n-r+1}{r} \cdot \dfrac{a}{x}$.

Therefore the r^{th} term is the greatest, when this factor is first less than unity.

$$\text{But if } \frac{n-r+1}{r} \cdot \frac{a}{x} < 1,$$

$$\text{then } na - ar + a < rx,$$

$$\text{or } na + a < rx + ar,$$

$$\text{or } rx + ar > na + a$$

$$\text{or } r > \frac{a(n+1)}{x+a}.$$

Therefore the factor $\dfrac{n-r+1}{r} \cdot \dfrac{a}{x}$ is first less than 1 if $r > \dfrac{a(n+1)}{x+a}$. Therefore the r^{th} term is the greatest when $r > \dfrac{a(n+1)}{x+a}$.

If $r=\dfrac{a(n+1)}{x+a}$, then $\dfrac{n-r+1}{r}\times\dfrac{a}{x}=1$. Therefore the r^{th} term is equal to the $(r+1)^{th}$, and each is greater than any other term.

180. All the terms of the expansion of $(x-a)^n$ are the same as those of $(x+a)^n$, except that the signs of all the even terms are negative.

$$\text{For, } (x-a)^n=\{x+(-a)\}^n$$
$$=x^n+n(-a)x^{n-1}+\frac{n(n-1)}{1\cdot 2}(-a)^2x^{n-2}+\text{\&c.}$$
$$=x^n-nax^{n-1}+\frac{n(n-1)}{1\cdot 2}a^2x^{n-2}-\ldots\ldots\ldots\text{\&c.}$$

281. To find the sum of the coefficients of the expansion of $(1+x)^n$.

$$(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+\ldots\ldots\ldots+nx^{n-1}+x^n.$$

Put 1 for x, then

$$(1+1)^n=2^n=1+n+\frac{n(n-1)}{2}+\ldots\ldots+n+1.$$

$\therefore$ the sum of the coefficients is 2^n.

282. *The sum of the coefficients of the even terms in the expansion of* $(1+x)^n$ *is equal to the sum of the coefficients of the odd terms.*

Put $x=-1$ in the expansion of $(1+x)^n$; then $(1-1)^n=0$
$$=1-n+\frac{n(n-1)}{1\cdot 2}-\frac{n(n-1)(n-2)}{2\cdot 3}+\text{\&c.}$$
$=$sum of the odd coefficients $-$ sum of the even coefficients.

Transposing the even coefficients to the other side, the sum of the even coefficients $=$ sum of the odd coefficients.

283. The trinomial $a+b+c$ may be raised to any power by considering two terms as one term, and proceeding as before.

$$\text{Thus, } (a+b+c)^n=\{a+(b+c)\}^n=a^n+n(b+c)a^{n-1}$$
$$+\frac{n(n-1)}{2}(b+c)^2a^{n-2}+\ldots\ldots$$

The powers of $b+c$ may be determined by the binomial theorem.

Examples worked out.

1. To expand $(3x+2y)^5$.

The expansion $=(3x)^5+5(3x)^4(2y)+\frac{5.4}{1\cdot 2}(3x)^3(2y)^2$

$$+\frac{5.4.3}{1.2.3}(3x)^2(2y)^3+\frac{5.4.3.2}{1.2.3.4}(3x)(2y)^4+(2y)^5$$

$$=243x^5+5\times 81x^4\times 2y+10\times 27x^3\times 4y^2$$
$$+10\times 9x^2\times 8y^3+5(3x)16y^4+32y^5$$
$$=243x^5+810x^4y+1080x^3y^2+720x^2y^3$$
$$+240xy^4+32y^5.$$

2. To expand $(2x-\frac{1}{2}y)^4$.

The expansion $=(2x)^4+4(2x)^3(-\frac{1}{2}y)+6(2x)^2(-\frac{1}{2}y)^2$

$$+4(2x)(-\tfrac{1}{2}y)^3+(-\tfrac{1}{2}y)^4$$
$$=16x^4-16x^3y+6x^2y^2-xy^3+\tfrac{1}{16}y^4.$$

3. To find the 9th term of $(a^2-x^2)^{12}$.

$$\text{9th term}=\frac{12.11.10.9.8.7.6.5}{1.2.3.4.5.6.7.8}(a^2)^{12-9+1}(-x^2)^{9-1}$$
$$=495a^8x^{16}.$$

4. To find the 1001st term of $(x^{\frac{1}{10}}+a^{\frac{1}{10}})^{1002}$

There are altogether $1002+1$ or 1003 terms.

The required term is 3rd from the end, therefore the coefficient of the required term = the coefficient of the 3rd term from the beginning.

$$\therefore 1001^{\text{th}}\text{ term}=\frac{1002\times 1001}{1.2}\times\left(x^{\frac{1}{10}}\right)^{1002-1001+1}\left(a^{\frac{1}{10}}\right)^{1001-1}$$
$$=501\times 1001x^{\frac{1}{5}}a^{100}.$$

5. To find the coefficient of x in the expansion of

$$\left(x^2+\frac{a^2}{x}\right)^5.$$

$$\left(x^2+\frac{a^2}{x}\right)^5=\left\{x^2\left(1+\frac{a^2}{x^3}\right)\right\}^5=x^{10}\left(1+\frac{a^2}{x^3}\right)^5.$$
$$x=x^{10}\div x^9=x^{10}\div(x^3)^3.$$

If A represent the coefficient of the term, then

$Ax^{10} \times \left(\frac{a^2}{x^3}\right)^3$ will be the term whose coefficient is

required. But $\left(\frac{a^2}{x^3}\right)^3 = \left(\frac{a^2}{x^3}\right)^{4-1}$.

$\therefore$ the 4th term is the term whose coefficient is required.

The 4th term $= \frac{5.4.3}{1.2.3} x^{10} \left(\frac{a^2}{x^3}\right)^3 = 10a^6x$. $\therefore$ etc.

6. If a be the sum of the odd terms and b the sum of the even terms in the expansion of $(1+x)^n$, prove that

$$a^2 - b^2 = (1-x^2)^n.$$

$$a + b = 1 + nx + \frac{n(n-1)}{2} x^2 + \ldots\ldots = (1+x)^n,$$

$$a - b = 1 - nx + \frac{n(n-1)}{2} x^2 - \ldots\ldots = (1-x)^n.$$

$$\therefore\ a^2 - b^2 = (1+x)^n (1-x)^n = (1-x^2)^n.$$

7. Simplify $\{1 + \sqrt{(x^2-1)}\}^5 + \{1 - \sqrt{(x^2-1)}\}^5$.

$$\because\ \{1+\sqrt{(x^2-1)}\}^5 = 1 + 5\sqrt{(x^2-1)} + 10(x^2-1)$$
$$+ 10(x^2-1)\sqrt{(x^2-1)} + 5(x^2-1)^2 + (x^2-1)^2\sqrt{(x^2-1)},$$

and $\{1-\sqrt{(x^2-1)}\}^5 = 1 - 5\sqrt{(x^2-1)} + 10(x^2-1)$
$$- 10(x^2-1)\sqrt{(x^2-1)} + 5(x^2-1)^2 - (x^2-1)^2\sqrt{(x^2-1)}.$$

$$\therefore\ \{1-\sqrt{(x^2-1)}\}^5 + \{1 - \sqrt{(x^2-1)}\}^5$$
$$= 2 + 20(x^2-1) + 10(x^2-1)^2$$
$$= 2 + 20x^2 - 20 + 10x^4 - 20x^2 + 10$$
$$= 10x^4 - 8.$$

8. Simplify $\{a + b\sqrt{(-1)}\}^6 + \{a - b\sqrt{(-1)}\}^6$.

The expression

$$= a^6 + 6a^5b\sqrt{(-1)} - 15a^4b^2 - 20a^3b^3\sqrt{(-1)} + 15a^2b^4$$
$$+ 6ab^5\sqrt{(-1)} - b^6$$
$$+ a^6 - 6a^5b\sqrt{(-1)} - 15a^4b^2 + 20a^3b^3\sqrt{(-1)} + 15a^2b^4$$
$$- 6ab^5\sqrt{(-1)} - b^6$$
$$= 2a^6 - 30a^4b^2 + 30a^2b^4 - 2b^6.$$

9. To find the binomial expansion of which the second, third and fourth terms are 192, 240 and 160 respectively.

Let $(x+a)^n$ be the expansion.

The second term $= nx^{n-1}a = 192.$ (A)

3rd term $= \dfrac{n(n-1)}{1.2} x^{n-2}a^2 = 240$ (B)

4th term $= \dfrac{n(n-1)(n-2)}{1.2.3} x^{n-3}a^3 = 160.$ (C)

Divide A by B and B by C, then

$$\frac{2}{n-1} xa^{-1} = \frac{192}{240} = \tfrac{4}{5}. \qquad (1).$$

and $$\frac{3}{n-2} xa^{-1} = \frac{240}{160} = \tfrac{3}{2}. \qquad (2).$$

Divide (1) by (2), $\dfrac{2(n-2)}{3(n-1)} = \frac{4}{5} \div \frac{3}{2} = \frac{8}{15}.$

$\therefore n = 6.$ (a)

From (1) $\frac{2}{5}xa^{-1} = \frac{4}{5}$; $\therefore xa^{-1} = 2.$

From (A) $6x^5a = 192$; $\therefore x^5a = 32$ (b)

Multiply (a) by (b), then $x^6 = 64 = 2^6$; $\therefore x = 2$ and $a = 1.$

Hence $(2+1)^6$ is the required expansion.

10. To shew that the middle term of the expansion of $\left(x+\dfrac{1}{x}\right)^{2n}$ is $= \dfrac{(2n-1)(2n-3)\ldots3.1}{\lfloor n} 2^n.$

The $(n+1)$th term is the middle term and it is

$$= \frac{2n(2n-1)\ldots\ldots(n+1)}{\lfloor n} x^n \left(\tfrac{1}{x}\right)^n = \frac{\lfloor 2n}{\lfloor n \lfloor n}$$

Of the $2n$ factors in $\lfloor 2n$, n factors are even and n factors odd;

$$\begin{aligned}\therefore \lfloor 2n &= 2n(2n-1)(2n-2)\ldots\ldots4.3.2.1\\ &= 2n(2n-2)\ldots6.4.2 \times (2n-1)(2n-3)\ldots7.5.3.1\\ &= 2^n n(n-1)(n-2)\ldots\ldots4.3.2.1 \times (2n-1)(2n-3)\ldots\ldots7.5.3.1\\ &= 2^n \lfloor n (2n-1)(2n-3)\ldots\ldots\ldots7.5.3.1.\end{aligned}$$

$$\therefore \text{the } (n+1)^{\text{th}} = \frac{(2n-1)(2n-3)\ldots3.1}{\lfloor n \lfloor n} 2^n \lfloor n = \&c.$$

11. If $(1+x)^n=p_0+p_1x+p_2x^2+p_3x^3+\dots\dots+p_nx^n$, find the value of $p_1+2p_2+3p_3+4p_4+\dots\dots+np_n$.

Put 1 for x, then

$$2^n=1+p_1+p_2+p_3+\dots\dots\dots\dots+p_n$$

$$\therefore 2^n=1+n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\dots\dots\dots+\frac{n(n-1)}{1.2}+n+1.$$

Put $n-1$ for n, then

$$2^{n-1}=1+(n-1)+\frac{(n-1)(n-2)}{1.2}+\frac{(n-1)(n-2)(n-3)}{1.2.3}+\dots$$
$$+\frac{(n-1)(n-2)}{1.2}+(n-1)+1$$

Multiply by n, then

$$2^n=n+n(n-1)+\frac{n(n-1)(n-2)}{1.2}+\dots\dots$$
$$+\frac{n(n-1)(n-2)}{1.2}+n(n-1)+n$$
$$=n+2\frac{n(n-1)}{2}+3\frac{n(n-1)(n-2)}{1.2.3}+\dots\dots+(n-2)\frac{n(n-1)}{1.2}$$
$$+(n-1)n+n$$
$$=p_1+2p_2+3p_3+4p_4+\dots\dots+(n-2)p_{n-2}+(n-1)p_{n-1}+np_n$$

$\therefore$ required value $=2^n$.

EXAMPLES LXXVI.

1. Expand $(1+2x)^{10}$.
2. Find the 4th term of $(5+4x)^{10}$.
3. Find the 16th term of $(\frac{1}{2}x+2y)^{30}$.
4. Find the 99th term of $(x-\frac{1}{x})^{100}$.
5. Find the 15th term of $(a^2-b^2)^{32}$.
6. Find the middle terms of $(x-x^{-1})^{17}$.
7. Find the two middle terms of $(1-x^{-1})^{24}$.
8. Find the 5th term of $(a^2-x^2)^{2n+1}$.
9. Find the $(r+1)$th term of $\left(a^2-\frac{1}{a^2}\right)^{2n}$.

10. Find the coefficient of x^{10} in the expansion of $(x+\frac{1}{x})^{20}$.

11. Find the coefficient of x^{11} in the expansion of $(x-\frac{1}{x})^{24}$.

12. Simplify $\{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}\}^6$
$+\{\sqrt{(a^2+x^2)}-\sqrt{(a^2-x^2)}\}^6$.

13. Simplify $\{a^2+x^2+(a^2-x^2)\sqrt{(-1)}\}^6$
$+\{a^2+x^2-(a^2-x^2)\sqrt{(-1)}\}^6$.

14. Simplify $\{1+\sqrt{(-3)}\}^8+\{1-\sqrt{(-3)}\}^8$.

15. Find the coefficient of x^{10}in the product of the expansions of $\{a^2+x^2\sqrt{(-1)}\}^{10}$ and $\{a^2-x^2\sqrt{(-1)}\}^{10}$.

16. Find the coefficient of x^8 in the expansion of $\{x^{-1}+x\sqrt{(-1)}\}^{20}$.

17. Simplify $\{a+b\sqrt{(-1)}\}^n+\{a-b\sqrt{(-1)}\}^n$.

18. Simplify $\dfrac{\{a+b\sqrt{(-1)}\}^n-\{a-b\sqrt{(-1)}\}^n}{\sqrt{(-1)}}$.

19. Find the coefficient of the middle term of $(1+x)^{4n+2}$.

20. If x be the sum of the odd terms of the expansion of $(a+b)^n$ and y the sum of the even terms, then prove that $x^2-y^2=(a^2-b^2)^n$.

21. If x be the sum of the odd terms of the expansion of $(a+b)^n$ and y the sum of the even terms, then prove that $4xy=(a+b)^{2n}-(a-b)^{2n}$.

22. Find the binomial expansion of which four consecutive terms are 135, 540, 1215 and 1458.

23. The coefficient of the $(r+1)$th term of the expansion of $(1+x)^n$ is equal to that of the $(r+3)$th term, shew that $2r=n-2$.

24. If a, b, c, d be the coefficients of any four consecutive terms of $(1+x)^n$, shew that

$$(ad+bc)(b-c)=2(ac^2-db^2).$$

Binomial Theorem any Exponent.

284. Before we proceed to prove the theorem universally we will investigate the following proposition :—

If $a+bx+cx^2+dx^3+\cdots=A+Bx+Cx^2+Dx^3+\ldots$ for all values of x, then $a=A$, $b=B$, $c=C$, $d=D$, &c.; that is, the coefficients of the like powers of x are equal.

Because x may have any value, let $x=0$, then $a=A$.

Again, taking away these equals from the original equation, we have $bx+cx^2+dx^3+\ldots=Bx+Cx^2+Dx^3\ldots\ldots$

Dividing by x,

$b+cx+dx^2+\ldots=B+Cx+Dx^2+\ldots\ldots$

Supposing $x=0$, we have $b=B$.

Likewise $c=C$, $d=D$, &c.

285. When n is a positive integer, we have proved that

$$(1+x)^n=1+nx+\frac{n(n-1)}{1.2}x^2+\ldots\ldots\ldots$$

We will now proceed to prove that this equation is true whatever be the value of n, that is, whether n is a positive fraction, as $\frac{1}{2}$, a negative integer, as -2, or a negative fraction, as $-\frac{2}{3}$.

Since the time of Sir Isaac Newton, who discovered this theorem, several demonstrations have been proposed. We will add here the proof devised by Euler.

Let us assume the equation

$$f(n)=1+nx+\frac{n(n-1)}{1.2}x^2+\ldots\ldots\ldots \qquad (1)$$

in which the symbol $f(n)$ means a function of n, whatever the value of n may be.

When n is a positive integer the right-hand member of the equation $=(1+x)^n$; $\therefore f(n)=(1+x)^n$. We shall now prove this for all values of n.

Let us assume another equation of the same form.

$$f(m)=1+mx+\frac{m(m-1)}{1.2}x^2+\ldots\ldots\ldots \qquad (2)$$

Multiplying we obtain an expression of the form

$$1+Ax+Bx^2+Cx^3+\ldots\ldots\ldots\ldots$$

$$\therefore f(n)\times f(m)=\left(1+nx+\frac{n(n-1)}{1.2}x^2+\ldots\right)$$

$$\times\left(1+mx+\frac{m(m-1)}{1.2}x^2+\ldots\right)$$

$$=1+(n+m)x+\left\{mn+\frac{n(n-1)}{1.2}+\frac{m(m-1)}{1.2}\right\}x^2+\ldots$$

Comparing this product with the assumed expression

$$1+Ax+Bx^2+Cx^3+\dots\dots\dots$$

we get $A=n+m$

$$B=nm+\frac{n(n-1)}{1.2}+\frac{m(m-1)}{1.2}+\dots$$

$$=\frac{(n+m)(n+m-1)}{1.2}.$$

Likewise $C=\dfrac{(n+m)(n+m-1)(n+m-2)}{1.2.3}.$

$$D=\frac{(n+m)(n+m-1)(n+m-2)(n+m-3)}{1.2.3.4}$$

$$\therefore f(n)\times f(m)=1+(n+m)x+\frac{(n+m)(n+m-1)}{1.2}x^2+\dots(A).$$

The expression (A) is formed by actual multiplication and it does not depend on the values of m and n, whether they be positive or negative, fractional or integral. If in the assumed equation (1) we put $m+n$ for n.

$$f(m+n)=1+(m+n)x+\frac{(m+n)(m+n-1)}{1.2}x^2+\dots\dots$$

Comparing this with (A) we find

$$f(m+n)=f(n)\times f(m) \quad \dots \quad \dots \quad \dots \quad \dots \quad (B).$$

Likewise $f(m+n+p)=f(n)\times f(m)\times f(p)$.

$\therefore$ generally $f(m+n+p+q+\dots)=f(n).f(m).f(p).f(q)\dots\dots\dots$

Let $m=n=p=q=\dfrac{r}{s}$, r and s being positive integers.

Then $f\left(\dfrac{r}{s}+\dfrac{r}{s}+\dots\text{to } s \text{ factors}\right)=f\left(\dfrac{r}{s}\right).f\left(\dfrac{r}{s}\right)\dots\text{to } s \text{ factors}$

$$\text{or } f(r)=\left\{f\left(\frac{r}{s}\right)\right\}^s.$$

$$\therefore (1+x)^r=\left\{f\left(\frac{r}{s}\right)\right\}^s.$$

$$\therefore (1+x)^{\frac{r}{s}}=f\left(\frac{r}{s}\right)$$

$$=1+\frac{r}{s}x+\frac{\frac{r}{s}\left(\frac{r}{s}-1\right)}{1.2}x^2+\dots$$

This proves the theorem when the index is a positive fraction.

Again, in equation (B) let $m=-n$, then

$$f(n)\times f(-n)=f(n-n)=f(0).$$

But the series $1+nx+\frac{n(n-1)}{1.2}x^2+\cdots$ becomes 1 when $n=0$;

$$\therefore f(0)=1.$$

$$\therefore f(n)\times f(-n)=1.$$

$$\therefore f(-n)=\frac{1}{f(n)}=\frac{1}{(1+x)^n}=(1+x)^{-n}.$$

$$\therefore (1+x)^{-n}=f(-n)$$

$$=1+(-n)x+\frac{-n(-n-1)}{1.2}x^2+\ldots\ldots$$

This proves the theorem when the index is a negative quantity.

286. We will add here another proof of the theorem.

Let $(1+x)^{\frac{n}{m}}=1+ax+bx^2+cx^3+\ldots\ldots\ldots$

then $(1+x)^n=(1+ax+bx^2+cx^3+\ldots)^m$.

$$\therefore 1+nx+\frac{n(n-1)}{1.2}x^2+\ldots=(1+ax)^m+mbx^2+\ldots\ldots$$

$$=1+max+\frac{m(m-1)}{2}a^2x^2+\ldots+mbx^2+\ldots$$

$$=1+max+\left(\frac{m(m-1)}{2}a^2+mb\right)x^2+\ldots$$

$$\therefore n=ma \text{ and } \frac{n(n-1)}{2}=\frac{m(m-1)}{2}a^2+mb. \qquad \text{(Art. 284)}$$

$$\therefore a=\frac{n}{m} \text{ and } b=\frac{n(n-1)}{2m}-\frac{m(m-1)}{2m}a^2$$

$$=\frac{n(n-1)}{2m}-\frac{m-1}{2}\times\frac{n^2}{m^2}$$

$$=\frac{n}{2m}\left\{n-1-\frac{(m-1)n}{m}\right\}$$

$$=\frac{n}{2m}\left\{\frac{n-m}{m}\right\}$$

$$=\frac{\frac{n}{m}\left(\frac{n}{m}-1\right)}{2}, \text{ etc. etc.}$$

$$\therefore\ (1+x)^{\frac{n}{m}}=1+\frac{n}{m}x+\frac{\frac{n}{m}\left(\frac{n}{m}-1\right)}{2}x^2+\dots\dots\dots$$

This proves the theorem when the exponent is a positive fraction.

$$\text{Again } (1+x)^{-n}=\frac{1}{(1+x)^n}=\frac{1}{1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\dots\dots}$$

$$=1-nx+\frac{n^2+n}{2}x^2+\dots\dots\dots$$

$$=1-nx+\frac{-n(-n-1)}{1.2}x^2+\dots\dots$$

This proves the theorem when the exponent is a negative quantity.

287. The $(r+1)^{\text{th}}$ term or the general term of the expansion of $(1+x)^n$ is

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{1\cdot 2\cdot 3\dots r}x^r.$$

The general term of the expansion of $(1+x)^{-n}$

$$=\frac{-n(-n-1)(-n-2)\cdots(-n-r+1)}{1\cdot 2\cdot 3\dots\dots\dots r}x^r.$$

$$=\frac{n(n+1)(n+2)\dots(n+r-1)}{1\cdot 2\cdot 3\dots r}(-1)^r x^r.$$

288. When n is a positive integer and is equal to r, the last term or the $(n+1)^{\text{th}}$ term is x^n. (Art. 277 Cor. 1)

When n is a fraction the series does not terminate, for r being a whole number, $n-r+1$, the last factor of the numerator, can never vanish.

Likewise, the series does not terminate when n is a negative quantity, for $-(n+r-1)$ can never vanish.

289. The $(r+1)^{\text{th}}$ term of the expansion of $(1-x)^{-n}$

$$=\frac{-n(-n-1)(-n-2)\dots(-n-r+1)}{1\cdot2\cdot3\dots\dots\dots r}(-x)^r$$

$$=\frac{n(n+1)(n+2)\dots(n+r-1)}{1\cdot2\cdot3\dots\dots r}(-1)^r(-1)^r x^r$$

$$=\frac{n(n+1)(n+2)\dots(n+r-1)}{1\cdot2\cdot3\dots\dots\dots r}x^r.$$

290. In the expansion of $(1+x)^n$ suppose $n=-1$, then

$$(1+x)^{-1}=1-x+x^2-x^3+x^4-\dots\dots\dots$$

If x be less than 1, each term of the expansion is less than the preceding one, and the series is called a *converging* series.

If x be greater than 1, each term is greater than the preceding term, and the series is called a *diverging* series.

291. By means of the Binomial Theorem roots of numbers may be approximately extracted. But the calculation must always be so arranged that the series, which the Binomial gives for the root required, may converge.

Ex. 1. To find $\sqrt{N}$ when N is very nearly a square.

Let $N=a^2+x$, then

$$\sqrt{N}=\sqrt{(a^2+x)}$$

$$=a\sqrt{\left(1+\frac{x}{a^2}\right)}=a\left(1+\frac{x}{a^2}\right)^{\frac{1}{2}}$$

$$=a\left(1+\tfrac{1}{2}\cdot\frac{x}{a^2}-\tfrac{1}{8}\cdot\frac{x^2}{a^4}\right)$$ nearly, for x is small and a^2 is very great.

Ex. 2. To find $\sqrt{(101)}$.

$$\sqrt{(101)}=\sqrt{(100+1)}=10(1+\tfrac{1}{100})^{\frac{1}{2}}$$

$$=10(1+\tfrac{1}{2}\times\tfrac{1}{100}-\tfrac{1}{8}\times\tfrac{1}{10000}+\cdots)$$

$$=10\cdot05-\cdot000125=10\cdot049875.$$

292. *To find the number of homogeneous products of* r *dimensions out of* m *letters* a, b, c, *&c., and their powers.*

By common division, or by the *Binomial Theorem,*

$$\frac{1}{1-ax}=1+ax+a^2x^2+a^3x^3+\ldots\ldots\ldots$$

$$\frac{1}{1-bx}=1+bx+b^2x^2+b^3x^3+\ldots\ldots\ldots$$

$$\frac{1}{1-cx}=1+cx+c^2x^2+c^3x^3\ldots\ldots\ldots$$

&c. = &c.

$$\therefore \frac{1}{1-ax}\times\frac{1}{1-bx}\times\frac{1}{1-cx}\times\text{\&c.}=1+(a+b+c+\text{\&c.})x$$
$$+(a^2+ab+ac+b^2+ac+bc+c^2+\ldots)x^2$$
$$+(a^3+a^2b+a^2c+ab^2+abc+\ldots\ldots\ldots)x^3$$
$$+\text{\&c. \&c.}$$

Let these be $=1+S_1x+S_2x^2+S_3x^3+\ldots\ldots\ldots$

$\therefore$ S_1 = sum of the quantities a, b, c, &c.;

S_2 = sum of all the products of a, b, c, &c., each of two dimensions and their powers.

S_3 = sum of the products, each of three dimensions, that can be formed of a, b, c, &c., and their powers, and so on.

S_r = sum of all the homogeneous products of a, b, c, &c and their powers, each of r dimensions.

Now to find the number of these products,
let $a=b=c=d=\text{\&c.}=1$.

Then $\frac{1}{1-ax}\times\frac{1}{1-bx}\times\frac{1}{1-cx}\times\cdots=\frac{1}{(1-x)^m}=(1-x)^{-m}$

$\therefore$ S_r = coefficient of x^r in the expansion of $(1-x)^{-m}$

$$=\frac{m(m+1)(m+2)\ldots(m+r-1)}{1.2.3\ldots\ldots\ldots m} \qquad \text{(Art. 289)}$$

Hence this is the number of homogeneous products of r dimensions that can be formed out of the m letters a, b, c, &c., and their powers.

293. To find the number of terms in the expansion of $(a+b+c+d+\text{\&c., to the } r\text{th term})^n$, when n is a positive integer.

The number of terms is the same as the number of homogeneous products of n dimensions that may be formed out of r things a, b, c, d, &c., and their powers, and it is

$$=\frac{r(r+1)(r+2)\dots(r+n-1)}{1\cdot2\cdot3\cdot4\dots n}.$$

Ex. To find the number of terms in the expansion of $(a+b+c)^n$.

$$\text{The number}=\frac{3\times4\times5\dots(n+2)}{1\cdot2\cdot3\dots n}$$

$$=\frac{(n+1)(n+2)}{1\cdot2}.$$

———o———

Examples worked out.

Ex. 1. To expand to four terms $(1+x)^{\frac{1}{2}}$.

$$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot2\cdot3}x^3+\dots$$

$$=1+\tfrac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\dots\dots$$

Ex. 2. To expand to four terms $(1-x)^{-\frac{1}{2}}$

$$(1-x)^{-\frac{1}{2}}=1-\tfrac{1}{2}(-x)+\frac{-\frac{1}{2}(-\frac{1}{2}-1)}{1\cdot2}(-x)^2.$$

$$+\frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1.2.3}(-x)^3$$

$$=1+\tfrac{1}{2}x+\tfrac{3}{8}x^2+\tfrac{5}{16}x^3+\dots$$

Ex. 3. To expand to four terms $(a^2-x^2)^{\frac{1}{3}}$.

$$(a^2-x^2)^{\frac{1}{3}}=\left\{a^2\left(1-\frac{x^2}{a^2}\right)\right\}^{\frac{1}{3}}=a^{\frac{2}{3}}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{3}}$$

$$=a^{\frac{2}{3}}\left\{1-\tfrac{1}{3}\frac{x^2}{a^2}+\frac{\frac{1}{3}(\frac{1}{3}-1)}{1.2}\left(-\frac{x^2}{a^2}\right)^2+\frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1\cdot2\cdot3}\left(-\frac{x^2}{a^2}\right)^3+\dots\right\}$$

$$=a^{\frac{2}{3}}\left\{1-\frac{x^2}{3a^2}-\frac{x^4}{9a^4}-\frac{5x^6}{81a^6}-\dots\right\}$$

$$=a^{\frac{2}{3}}-\frac{x^2}{3a^{\frac{4}{3}}}-\frac{x^4}{9a^{\frac{10}{3}}}-\frac{5x^6}{81a^{\frac{16}{3}}}-\dots.$$

Ex. 4. To find the sixth term of $(1+x)^{\frac{5}{2}}$.

$$\text{Sixth term} = \frac{\frac{5}{2} \times (\frac{5}{2}-1)(\frac{5}{2}-2)(\frac{5}{2}-3)(\frac{5}{2}-4)}{1\cdot2\cdot3\cdot4\cdot5}x^5$$

$$= \frac{1}{2^5} \times \frac{5\cdot3\cdot1\cdot(-1)\cdot(-3)}{1\cdot2\cdot3\cdot4\cdot5}x^5$$

$$= \frac{3}{256}x^5.$$

Ex. 5. To find the $(r+1)^{\text{th}}$ term of $(1-x)^{-2}$.

$$(r+1)^{\text{th}}\text{ term} = \frac{-2(-2-1)(-2-2)\ldots(-2-r+1)}{1\cdot2\cdot3\cdot4\ldots r}(-x)^r$$

$$= \frac{(-1)^r 2\cdot3\cdot4\ldots\ldots\ldots(r+1)}{1\cdot2\cdot3\cdot4\ldots r}(-x)^r$$

$$= \frac{2\cdot3\cdot4\ldots\ldots\ldots(r+1)}{1\cdot2\cdot3\cdot4\ldots r}x^r.$$

$$= (r+1)x^r.$$

Ex. 6. To find the $(r+1)^{\text{th}}$ term of $\frac{1}{\sqrt[4]{(1-x)}}$.

$$\frac{1}{\sqrt[4]{(1-x)}} = \frac{1}{(1-x)^{\frac{1}{4}}} = (1-x)^{-\frac{1}{4}}$$

$$\text{The }(r+1)^{\text{th}}\text{ term} = \frac{-\frac{1}{4}(-\frac{1}{4}-1)(-\frac{1}{4}-2)\ldots(-\frac{1}{4}-r+1)}{1\cdot2\cdot3\cdot4\ldots\ldots\ldots r}(-x)^r$$

$$= \frac{(-1)^r\frac{1}{4}\times\frac{5}{4}\times\frac{9}{4}\times\ldots\times(\frac{4r-3}{4})}{1\cdot2\cdot3\ldots\ldots\ldots r}(-x)^r$$

$$= \frac{1\cdot5\cdot9\ldots(4r-3)}{4^r\lfloor r}x^r.$$

Ex. 7. To find the coefficient of x^{4r} in the expansion of $(1+x^2)^{-3}$.

$(x^2)^{2r}=x^{4r}$. The coefficient of the $(2r+1)^{\text{th}}$ term is the required coefficient.

$$(2r+1)^{\text{th}} \text{ term} = \frac{-3 \times -4 \times -5 \times \ldots \times (-3-2r+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots 2r} x^{4r}$$

$$= \frac{(-1)^{2r} 3 \cdot 4 \cdot 5 \ldots (2r+2)}{1 \cdot 2 \cdot 3 \ldots 2r} x^{4r}$$

$$= \frac{(2r+1)(2r+2)}{2} x^{4r}$$

$$= (2r+1)(r+1) x^{4r}.$$

Ex. 8. To find the coefficient of x^r in the expansion of $(1+x+x^2+x^3+\cdots\text{to infinity})^3$.

$$1+x+x^2+x^3 \cdots \text{to infinity} = \frac{1}{1-x}.$$

$$\therefore\ (1+x+x^2 \ldots\ldots \text{to infinity})^3 = (1-x)^{-3}.$$

$$(r+1)^{\text{th}} \text{ term} = \frac{-3 \times -4 \times -5 \times \ldots (-3-r+1)}{1 \cdot 2 \cdot 3 \ldots\ldots r} (-x)^r$$

$$= \frac{3 \cdot 4 \cdot 5 \cdot 6 \ldots\ldots (r+2)}{1 \cdot 2 \cdot 3 \cdots r} x^r$$

$$= \tfrac{1}{2}(r+1)(r+2) x^r.$$

$\therefore$ the required coefficient is $\frac{1}{2}(r+1)(r+2)$.

Ex. 9. To find the coefficient of x^{2n} in the expansion of $\sqrt{\left(\frac{1+x}{1-x}\right)}$ in a series of ascending powers of x.

$$\sqrt{\left(\frac{1+x}{1-x}\right)} = \sqrt{\left(\frac{1+x}{1-x} \times \frac{1+x}{1+x}\right)} = \frac{1+x}{\sqrt{(1-x^2)}} = (1+x)(1-x^2)^{-\frac{1}{2}}$$

$(n+1)^{\text{th}}$ term of $(1-x^2)^{-\frac{1}{2}}$

$$= \frac{-\frac{1}{2} \times -\frac{3}{2} \times -\frac{5}{2} \times \ldots\ldots (-\frac{1}{2}-n+1)}{1 \cdot 2 \cdot 3 \ldots\ldots n} (-x^2)^n$$

$$= \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)}{1 \cdot 2 \cdot 3 \ldots\ldots n \times 2^n} x^{2n}.$$

$$n^{\text{th}} \text{ term of } (1-x^2)^{-\frac{1}{2}} = \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-3)}{2^{n-1} \times \underline{|n-1}} x^{2n-2}.$$

$\therefore$ the coefficient of the $(n+1)^{\text{th}}$ term of $(1-x^2)^{-\frac{1}{2}}$ is the required coefficient.

Ex. 10. To find the number of terms in the expansion of $(a+b+c+d)^8$.

$$\text{The number}=\frac{4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11}{1\cdot2\cdot3\cdot4\cdot5\;6\cdot7\cdot8}=\frac{9\cdot10\cdot11}{1\cdot2\cdot3}=165.$$

Ex. 11. Shew that

$$\sqrt{\left(\frac{1+x}{1-x}\right)}=1+\frac{x}{1+x}+\frac{3}{2}\left(\frac{x}{1+x}\right)^2+\frac{5}{2}\left(\frac{x}{1+x}\right)^3+\ldots$$

$$\sqrt{\left(\frac{1+x}{1-x}\right)}=\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}}=\left(1-\frac{2x}{1+x}\right)^{-\frac{1}{2}}$$

$$=1+\frac{x}{1+x}+\frac{3}{2}\left(\frac{x}{1+x}\right)^2+\ldots$$

Ex. 12. To find the sum of the squares of the coefficients of $(1+x)^n$.

Let $(1+x)^n=1+nx+Ax^2+\ldots+Ax^{n-2}+nx^{n-1}+x^n$

then $(1+x)^n=x^n+nx^{n-1}+Ax^{n-2}+\ldots+Ax^2+nx+1$.

$\therefore\ (1+x)^{2n}=x^n+n^2x^n+A^2x^n+\ldots\ldots+A^2x^n+n^2x^n+x^n+\&c.$

$\therefore$ the sum of the squares of the coefficients of $(1+x)^n$ is the coefficient of x^n in the expansion of $(1+x)^{2n}$.

But the coefficient of x^n in the expansion of $(1+x)^{2n}$ is the coefficient of the $(n+1)th$ term.

$\therefore\ (n+1)^{\text{th}}$ coefficient

$$=\frac{2n(2n-1)(2n-2)\ldots\ldots\ldots\{2n-n+1\}}{1\cdot2\cdot3\cdot4\ldots n}$$

$$=\frac{2n(2n-1)(2n-2)\cdots\{n+1\}n(n-1)\ldots3\cdot2\cdot1}{(1\cdot2\cdot3\cdot4\ldots n)^2}$$

$$=\frac{2n(2n-2)(2n-4)\ldots4\cdot2\times(2n-1)(2n-3)\ldots3\cdot1}{(1\cdot2\cdot3\ldots n)^2}$$

$$=\frac{2^n(2n-1)(2n-3)\ldots3\cdot1\times n(n-1)(n-2)\cdots3\cdot2\cdot1}{(1\cdot2\cdot3\ldots n)^2}$$

$$=\frac{2^n(2n-1)(2n-3)\ldots3\;1}{1\cdot2\cdot3\cdot4\ldots n}.$$

EXAMPLES LXXVII.

Find the first four terms of the expansions of the twelve following expressions.

1. $(1+x)^{\frac{1}{4}}$. 2. $(1-x)^{\frac{2}{3}}$. 3. $(1-x)^{-2}$.

4. $(1+x)^{-n}$. 5. $(1-x)^{-\frac{3}{4}}$. 6. $(1-x)^{-\frac{1}{2}}$.

7. $(8a-3x)^{\frac{1}{3}}$. 8. $(a^2-xy)^{-\frac{3}{4}}$. 9. $(1-5x^2)^{-n}$.

10. $(2a-3x)^{\frac{3}{4}}$. 11. $(a^2-b^2y)^{-\frac{1}{4}}$. 12. $(1-3x)^{\frac{13}{4}}$.

Find the $(r+1)$th term in the expansions of the following expressions.

13. $(1+x)^{-3}$. 14. $(1-x)^{-4}$. 15. $(1-x)^{-n}$.

16. $(1-ax^2)^{-\frac{2}{3}}$. 17. $(1-x^2)^{\frac{1}{n}}$. 18. $\dfrac{1}{\sqrt[3]{(1+x)}}$.

19. $\dfrac{x^2}{\sqrt[3]{(1-x)}}$. 20. $\left(\dfrac{a^{10}}{1-x^5}\right)^{\frac{2}{3}}$. 21. $\left(\dfrac{a-2b}{x^2}\right)^{-\frac{1}{3}}$.

22. Find the $(2r+1)$th term of $(1-\sqrt{x})^{-4}$.

23. Find the coefficient of x^{20} in the expansion of $\sqrt[3]{(1-x^2)}$.

24. Find the tenth term of the expansion of $(a^2+2x^2)^{-2}$.

Apply the Binomial Theorem to find to 4 places of decimals the roots of the following :—

25. $\sqrt[5]{(35)}$. 26. $\sqrt[3]{(999)}$. 27. $\sqrt[5]{(100008)}$. 28. $\sqrt[6]{(65 \cdot 28)}$.

29. If m and n be the general terms of the expansions of

$$\frac{1}{\sqrt{(1-x)}} \text{ and } \left\{\frac{1}{\sqrt{(1-x)}}\right\}^3, \text{ shew that } n=(2r+1)m.$$

30. If $(1-x)^{-\frac{3}{2}}=1+Ax+Bx^2+Cx^3+\ldots\ldots$, find the values of A, B, C.

31. Find the 10th term of the expansion of $\sqrt{(2ax-x^2)}$.

32. Find the coefficient of x^n in the expansion of $\dfrac{(1+x)^2}{(1-x)^3}$.

33. Find the coefficient of x^{3n} in the expansion of

$$\sqrt{\left\{\frac{1-x}{1+x+x^2}\right\}}.$$

34. Find the coefficient of x^{10} in the expansion of

$$(1+x+x^2)^{-2}.$$

35. Find the coefficient of x^{2n} in the expansion of

$$\frac{1-x}{(1+x)^2}.$$

36. Find the number of terms in the expansion of

$$(a+b+c+d)^{12}.$$

37. Find the sum of the squares of the coefficients of x^n in the expansion of $(1-x)^{-2}$.

38. Find the coefficient of x^r in the expansion of $\left\{\frac{a+x}{a-x}\right\}^{\frac{3}{2}}$.

39. Find the greatest coefficient in the expansion of $(1+\frac{2x}{3})^{\frac{1}{2}}$.

40. Find the greatest coefficient in the expansion of

$(1-\frac{2}{3}x)^{-\frac{1}{4}}$.

41. Find the greatest term in the expansion of $(1+\frac{4}{3})^{\frac{5}{3}}$.

42. Find the coefficient of x^{12} in the expansion of

$(x^2+x-2)^{-1}$.

43. The coefficient of what term of $(1-x)^{\frac{1}{2}}$ is $\frac{1}{11}$ of the coefficient of the same term of $(1+x)^{-\frac{1}{2}}$?

44. Find the coefficient of a certain term in $(1+x)^{-\frac{1}{2}}$ which is $(2n+1)$ times the coefficient of the same term in $(1-x)^{\frac{1}{2}}$.

45. Prove that

$$\sqrt{\left(\frac{a-x}{a+x}\right)}=1-\frac{x}{a-x}+\frac{3}{2}\left(\frac{x}{a-x}\right)^2-\frac{5}{2}\left(\frac{x}{a-x}\right)^3+\ \ldots\ldots$$

46. *Find the coefficient of x^n in the expansion of*

$$(1+2x+3x^2+4x^3+\ldots \text{ to infinity})^2.$$

47. Prove that the coefficient of the $(n+1)$th term of $(1-2x)^{-\frac{1}{2}}$ is double the coefficient of x^n in $(1-\frac{1}{2}x)^{-n}$.

Logarithms.

294. If n be a number such that $n=a^x$, then x is said to be the logarithm of n, and a is called the base of the system of logarithms.

Hence a logarithm may be defined to be the index or power to which the base is to be raised that the result may be equal to a given number.

The logarithm of n to the base a is written $\log_a n$; $\log_a n=x$ expresses the same relation as $a^x=n$.

Ex. $2^5=32$. $\therefore$ 5 is the logarithm of 32 to the base 2.

295. $\because a^0=1$, therefore the logarithm of 1 is 0 whatever the base may be. $\because a^1=a$, therefore the logarithm of the base itself is unity.

296. In $a^x=n$, a may be assumed at pleasure; and for every different value so assumed, a different system of logarithms will be formed. In the common tabular logarithms, a is 10, and consequently, 0, 1, 2, 3...x, are the logarithms of 1, 10, 100, 1000,...$(10)^x$.

297. Since the tabular logarithm of 10 is 1 and of 1 is 0, therefore the logarithm of a number between 1 and 10 is less than 1; and in the same manner, the logarithm of a number between 10 and 100 is between 1 and 2; of a number between 100 and 1000, is between 2 and 3; &c.

These logarithms are also real quantities, to which approximation, sufficiently accurate for all practical purposes may be made.

Thus, if x be the logarithm of 5, then $(10)^x=5$; let $\frac{2}{3}$ be substituted for x, and $10^{\frac{2}{3}}$ is found to be less than 5, therefore $\frac{2}{3}$ is less than the logarithm of 5; but $10^{\frac{3}{4}}$ is greater than 5, or $\frac{3}{4}$ is greater than the logarithm of 5; the value set down in the tables is $\cdot 69897$, and $10^{\cdot 69897}=5$ *nearly*.

298. *The sum of the logarithms of two numbers is the logarithm of their product; and the difference of the logarithms is the logarithm of their quotient.*

Let $x = \log_a c$, $y = \log_a d$.

then $a^x = c$ and $a^y = d$.

$$a^{x+y} = cd, \text{ and } a^{x-y} = \frac{c}{d}.$$

$$\therefore \log_a(cd) = x + y = \log_a c + \log_a d,$$

$$\text{and } \log_a\left(\frac{c}{d}\right) = x - y = \log_a c - \log_a d.$$

Ex. 1. $\operatorname{Log}_a(3 \times 7) = \log_a 3 + \log_a 7.$

Ex. 2. $\operatorname{Log}_a(pqr) = \log_a(pq) + \log_a r.$
$= \log_a p + \log_a q + \log_a r.$

Ex. 3. $\operatorname{Log}_a \frac{5}{7} = \log_a 5 - \log_a 7.$

Ex. 4. $\operatorname{Log}_{10} 600 = \log_{10} 6 + \log_{10} 100$
$= \log_{10} 6 + 2.$

Obs. $\operatorname{Log}_{10} n$ is generally written $\log n$.

Ex. 5. $\operatorname{Log} \cdot 05 = \log \frac{5}{100} = \log 5 - \log 100 = \log 5 - 2$; also $= \log \frac{10}{2} - 2 = \log 10 - \log 2 - 2 = -\log 2 - 1.$

Or $\log \cdot 05 = \log \frac{1}{20} = -\log 20 = -\log 2 - \log 10$
$= -\log 2 - 1.$

299. *If the logarithm of any number be multiplied by n, the product is the logarithm of that number raised to the* nth *power.*

Let d be the number whose logarithm is x, or $a^x = d$; then $a^{nx} = d^n$; that is nx is the logarithm of d^n;

$$\therefore n \log d = \log d^n.$$

Ex. $5 \log 13 = \log (13)^5.$

330. *If the logarithm of a number be divided by n, the quotient is the logarithm of the* nth *root of that number.*

Let $a^x = d$, then $a^{\frac{x}{n}} = d^{\frac{1}{n}}$, $\frac{x}{n} = \log_a d^{\frac{1}{n}}$ or $\frac{\log_a d}{n} = \log_a d^{\frac{1}{n}}$.

Ex. $\frac{1}{4} \log 5 = \log (5^{\frac{1}{4}}).$

301. To find the relation between the logarithms of the same number to different bases.

Let $x=\log_a n$, $y=\log_b n$.

then $a^x=n$, $b^y=n$.

$\therefore\ a^x=b^y$.

$\therefore\ a^{\frac{x}{y}}=b$, and $a=b^{\frac{y}{x}}$

$\therefore\ \frac{x}{y}=\log_a b$ and $\frac{y}{x}=\log_b a$.

$\therefore\ x=y\ \log_a b=\frac{y}{\log_b a}$.

Cor. $\log_a b\times\log_b a=\frac{x}{y}\times\frac{y}{x}=1$.

302. $10^{1\cdot20412}=16$. Here the integral part of $1\cdot20412$ is called the *characteristic*, and the decimal part the *mantissa*.

303. In the common system of logarithms, if the logarithm of any number be known, we may easily find out the logarithm of the product or quotient of that number by any power of 10.

For if we have $\log N=a$.

$\text{Log}\,(N\times10^n)=\log N+\log 10^n=\log N+n$.

$$\text{Log}\left(\frac{N}{10^n}\right)=\log N-\log 10^n=\log N-n.$$

304. If the number have n digits before the decimal point, the characteristic is $n-1$, and if the number be a decimal with $n-1$ cyphers before the first significant figure, the characteristic is $-n$.

Ex. Suppose we know that $\log 8\cdot64=\cdot9365137$

$$\begin{aligned}\text{Log}\,864&=\log(8\cdot64\times100)\\&=\log 8\cdot64+\log 100\\&=\log 8\cdot64+2\\&=2\cdot9365137.\end{aligned}$$

Likewise $\log 8640=3\cdot9365137$.

$$\begin{aligned}\text{Again }\log\ \cdot864&=\log(8\cdot64\div10)\\&=\log 8\cdot64-1\\&=\cdot9365137-1.\end{aligned}$$

Obs. $\cdot 9365137 - 1$ is generally written $\bar{1}\cdot 9365137$, the mantissa being always positive.

Likewise $\log \cdot 00864 = \bar{3}\cdot 9365137$.

305. The utility of a table of logarithms in arithmetical calculations will from hence be manifest; the multiplication and division of numbers being performed by the addition and subtraction of these artificial representatives; and the involution or evolution of numbers by multiplying or dividing their logarithms by the indices of the powers or roots required.

Ex. Let the value of $\sqrt[5]{\{7\sqrt{2} \times \sqrt[3]{(3)}\}}$ be required.

Given $\log 7 = \cdot 845098$; $\log 2 = \cdot 301030$, $\log 3 = \cdot 477121$; $\log 1\cdot 70188 = \cdot 2309306$.

$$\begin{aligned} &\text{Log} \sqrt[5]{\{7 \times \sqrt{2} \times \sqrt[3]{(3)}\}} \\ &= \tfrac{1}{5}\{\log 7 + \tfrac{1}{2}\log 2 + \tfrac{1}{3}\log 3\} \\ &= \tfrac{1}{5}\{\cdot 845098 + \cdot 150515 + \cdot 1590404\} \\ &= \cdot 2309306. \end{aligned}$$

But $\log 1\cdot 70188 = \cdot 2309306$.

$$\therefore \sqrt[5]{\{7 \times \sqrt{2} \times \sqrt[3]{(3)}\}} = \cdot 170188.$$

Examples worked out.

Ex. 1. To find the logarithm of 64 to the base $2\sqrt{2}$.

$$\text{Let } (2\sqrt{2})^x = 64. \quad \therefore 2^{\frac{3}{2}x} = 64.$$

$$\therefore \log 2^{\frac{3}{2}x} = \log 64 = \log 2^6.$$

$$\therefore \tfrac{3}{2}x \log 2 = 6 \log 2$$

$$\therefore \tfrac{3}{2}x = 6\,;\ x = 4.$$

Ex. 2. To find the characteristic of the logarithm of 25 to the base 3.

$$3^3 = 27 \text{ and } 3^2 = 9.$$

$\therefore \log_3 25$ lies between 2 and 3. $\therefore$ 2 is the characteristic of $\log_3 25$.

Ex. 3. To find the characteristic of $\log_{10}1120$ and of $\log_{10}\cdot0112$.

$10^3=1000, 10^4=10000$. $\therefore$ 3 is the characteristic of $\log_{10}1120$.

$\text{Log}_{10}.0112=\log(112\div10000)=\log112-4$.

2 is the characteristic of log 112. $2-4$ or -2 is the characteristic of $\log_{10}.0112$.

Ex. 4. Given $\log2=\cdot30103000$, $\log3=\cdot4771213$, to find the logarithms of 5, 25, of 7·2 and of ·96.

$$\text{Log}\,5=\log\tfrac{10}{2}=1-\log2=1-\cdot3010300=\cdot6989700.$$

$$\begin{aligned}\text{Log}\,25&=\log\tfrac{100}{4}=\log100-\log4\\&=2-2\log2=2-\cdot6020600\\&=1\cdot3979400.\end{aligned}$$

$$\begin{aligned}\text{Log}\,7\cdot2&=\log\frac{9\times8}{10}\\&=\log9+\log8-1=2\log3+3\log2-1\\&=\cdot9542425+\cdot9030900-1\\&=\cdot8573325.\end{aligned}$$

$$\begin{aligned}\text{Log}\,\cdot96&=\log\frac{3\times32}{100}=\log3+\log32-\log100\\&=\log3+5\log2-2\\&=\cdot4771213+1\cdot5051500-2\\&=\cdot9822712-1=\bar{1}\cdot9822712.\end{aligned}$$

Ex. 5. Given $\log108=2\cdot0334238$, $\log324=2\cdot5105450$, to find log 2 and log 3.

$$\text{Log}\,108=\log(3^3\times2^2)=2\log2+3\log3=2\cdot0334238,$$

$$\text{Log}\,324=\log(3^4\times2^2)=2\log2+4\log3=2\cdot5105450.$$

$$\begin{aligned}\therefore\ \log3&=2\cdot5105450-2\cdot0334238\\&=\cdot4771212.\end{aligned}$$

$$\begin{aligned}2\log2&=2\cdot0334238-3\log3\\&=\cdot6020600.\end{aligned}$$

Ex. 6. Given $\log 2 = \cdot 3010300$, find $\log \sqrt[3]{(\cdot 00004)}$.

$$\log \sqrt[3]{(\cdot 00004)} = \tfrac{1}{3}\log(\cdot 00004)$$
$$= \tfrac{1}{3}(\log 2 - 5) = \tfrac{1}{3}(\cdot 3010300 - 5)$$
$$= \tfrac{1}{3}(1\cdot 3010300 - 6)^{\circ}$$
$$= \cdot 4336766 - 2 = \bar{2}\cdot 4336766.$$

° Obs. To divide $(\cdot 3010300 - 5)$ by 3, the characteristic must be so increased that it may be divisible by 3.

Ex. 7. Given $\log 2 = \cdot 3010300$, to find

$$\log \sqrt[5]{\{\cdot 0032 \times \sqrt{(\cdot 00625)} \times \sqrt[3]{(\cdot 0002)}\}}.$$

The required logarithm

$$= \tfrac{1}{5}\{\log \cdot 0032 + \tfrac{1}{2}\log \cdot 00625 + \tfrac{1}{3}\log \cdot 0002\}$$
$$= \tfrac{1}{5}\{(5\log 2 - 4) + \tfrac{1}{2}(\log 5^4 - 5) + \tfrac{1}{3}(\log 2 - 4)\}$$
$$= \tfrac{1}{5}\{(1\cdot 5051500 - 4) + \tfrac{1}{2}(2\cdot 7958800 - 5) + \tfrac{1}{3}(\cdot 3010300 - 4)\}$$
$$= \tfrac{1}{5}\{\bar{3}\cdot 5051500 + \tfrac{1}{2}(\bar{3}\cdot 7958800) + \tfrac{1}{3}(\bar{4}\cdot 3010300)\}$$
$$= \tfrac{1}{5}\{\bar{3}\cdot 5051500 + \tfrac{1}{2}(-4 + 1\cdot 7958800) + \tfrac{1}{3}(-6 + 2\cdot 3010300)\}$$
$$= \tfrac{1}{5}\{\bar{3}\cdot 5051500 - 2 + \cdot 8979400 + -2 + \cdot 7670100\}$$
$$= \tfrac{1}{5}\{\bar{7} + \cdot 5051500 + \cdot 8979400 + \cdot 7670100\}$$
$$= \tfrac{1}{5}\{\bar{5} + \cdot 1701000\}$$
$$= 1\cdot 0340200.$$

Ex. 8. Given $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$,

$\log 7 = \cdot 8450983$, $\log 715\cdot 7816 = 2\cdot 8547823$,

find $\sqrt{\{5 \times \sqrt{(21)}\sqrt{5}\}}$.

$$\text{Log } 5 = \log \tfrac{10}{2} = \log 10 - \log 2 = 1 - \cdot 3010300 = \cdot 6989700.$$

$$\text{Log } \sqrt{\{5 \times \sqrt{(21)}\sqrt{5}\}} = \tfrac{1}{2}\{\log 5 + \tfrac{1}{2}\log 21 + \tfrac{1}{2}\log 5\}$$
$$= \tfrac{1}{2}\{\log 5 + \tfrac{1}{2}\log 3 + \tfrac{1}{2}\log 7 + \tfrac{1}{2}\log 5\}$$
$$= \tfrac{1}{2}\{\cdot 6989700 + \cdot 2385606 + \cdot 4225491 + \cdot 3494850\}$$
$$= \cdot 8547823.$$

But $\log 7\cdot 157816 = \cdot 8547823$.

$$\therefore \sqrt{\{5 \times \sqrt{(21)}\sqrt{5}\}} = 7\cdot 157816.$$

Ex. 9. Given $(8192)^x = \dfrac{4}{32^x}$ find x.

$$2^{13x} = 2^2 \times 2^{-5x} = 2^{2-5x}.$$

$$13x \log 2 = (2-5x) \log 2.$$

$$\therefore \; 13x = 2 - 5x,$$

$$18x = 2,$$

$$x = \tfrac{1}{9}.$$

Ex. 10. $\dfrac{8^x}{4^{x+y}} = 16$, and $x = 4y$, find x and y.

$$8^x = 16 \times 4^{x+y};$$

$$2^{3x} = 2^4 \times 2^{2x+2y} = 2^{4+2x+2y}.$$

$$\therefore \; 3x \log 2 = (4 + 2x + 2y) \log 2.$$

$$\therefore \; 3x = 4 + 2x + 2y$$

$$\therefore \; x - 2y = 4$$

$$4y - 2y = 2y = 4; \quad \therefore \; y = 2, \; x = 8.$$

EXAMPLES LXXVIII.

1. Find the logarithm of 32 to the base $\sqrt{2}$.

2. Find the logarithm of 324 to the base $3\sqrt{2}$.

3. Given $\log 2 = \cdot 3010300$, find the logarithms of 5, 16, 32.

4. Given $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$, find $\log 12$, $\log 15$, $\log 75$ and $\log 1\cdot 8$.

5. Find the characteristic of the logarithm of 15 to the base 2.

6. Find the characteristic of the logarithm of 32 to the base 4.

7. In the common system of logarithms, find the characteristics of the logarithms of 1023, 45067, 143·045 and of ·044.

8. Given $\log 2$ and $\log 3$ (see Ex. 4), find $\log 2160$ and $\log \sqrt[5]{(\cdot 0075)}$.

9. Given $\log 162 = 2\cdot2095150$, $\log 216 = 2\cdot3344538$, find $\log 2$ and $\log 3$.

10. Given $\log 324 = 2\cdot5105450$, $\log 375 = 2\cdot5740313$, find $\log 3$ and $\log 5$.

11. Given $\log 2$ and $\log 3$, find the value of

$$\log \sqrt[5]{\left\{\frac{(135)^2 \times \sqrt{3} \times \sqrt[3]{(15)}}{40 \times 81}\right\}}.$$

12. Given $\log 2$, $\log 3$, find $\log \sqrt{(\cdot00006)}$ and $\log \sqrt[5]{(\cdot0384)}$.

13. Find the number of digits in $(13)^{100}$, given $\log 13 = 1\cdot1139434$.

14. Given $\log 2$, find $\log \sqrt[3]{\{(\frac{32}{5})^4\}}$ and $\log \sqrt[5]{\{(\frac{125}{4})^3\}}$.

15. $2^{3x} \times 6^{8x-5} = 3^{2(x-1)} \times 2^{5x-2}$, find x.

16. Given $\log 2$ and $\log 3$, and $3^x = 25$, find x.

17. $\text{Log} \sqrt{\frac{125}{128}} = \bar{1}\cdot99485$, find $\log 2$.

18. Find the logarithm of 512 to the base 4.

19. Given $\log 2$ and $\log 3$, find the value of

$$\sqrt[5]{\left\{\frac{5\sqrt[3]{(432)}}{\sqrt[5]{(\cdot01)}}\right\}}, \quad \log 2\cdot48595 = \cdot3954930.$$

20. Given $\log 2 = \cdot3010300$, $\log 3 = \cdot4771213$, and

$\log 259569 = 5\cdot4142524$,

$$\text{find } \sqrt[5]{\left\{\frac{(\cdot32)^8 \times (625)^4}{(\cdot00432)^2 \times (\cdot3125)^3 \times 25}\right\}}.$$

21. Given $\log 437437 = 5\cdot6409153$, find the value of

$$\sqrt[5]{\left\{\frac{\sqrt[3]{(\cdot0125)} \times \sqrt{(31\cdot25)}}{\cdot00081}\right\}}.$$

22. Given $\log 2 = \cdot3010300$ and $\log 7 = \cdot8450980$, find the value of x to two decimal places when $10^{\frac{1}{x}} = 2\cdot45$.

23. Given $\log 2 = \cdot3010300$ and $\log 5444525 = 6\cdot7359600$, find the value of $(2\cdot56)^{\frac{3}{2}} \times (2\cdot56)^2 \times (2\cdot56)^{\frac{5}{2}} \times \cdots$ to six factors.

24. Given log 2 and log 3, find to two places of decimals the *real* value of x in the equation $2^x - 6 \times 2^{-x} = 1$.

25. Given $\log 101 = 2{\cdot}0043214$, $\log 111{\cdot}5675 = 2{\cdot}0475354$, find the sum of

$$\frac{101}{100} + \left(\frac{101}{100}\right)^2 + \left(\frac{101}{100}\right)^3 + \text{\&c. to ten terms.}$$

26. If the number of births in a year be $\frac{1}{48}$ of the population at the beginning of the year, and the number of deaths $\frac{1}{60}$; find in what time the population will be doubled; having given, $\log 2 = {\cdot}3010300$, $\log 241 = 2{\cdot}3820170$ and $\log 240 = 2{\cdot}3802112$.

Exponential and Logarithmic Series.

EXPONENTIAL THEOREM.

306. *To expand a number in a series of ascending powers of its logarithm to a given base.*

Let the given number be such that it is equal to a^x, then to find the expansion of the given number in a series of ascending powers of x.

$$a^x = \{1 + (a-1)\}^x = (1+c)^x, \text{ where } c = a - 1.$$

$$\text{But } (1+c)^x = 1 + cx + x\frac{x-1}{2}c^2 + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}c^3 + \ldots\ldots$$

$$= 1 + cx + \frac{x^2 - x}{2}c^2 + \frac{x^3 - 3x^2 + 2x}{2 \cdot 3}c^3 + \ldots\ldots$$

$$= 1 + x(c - \tfrac{1}{2}c^2 + \tfrac{1}{3}c^3 - \text{\&c.}) + x^2(\tfrac{1}{2}c^2 - \tfrac{1}{2}c^3 + \text{\&c.}) + \text{\&c.}$$

$$= 1 + nx + n_2x^2 + n_3x^3 + \ldots\ldots$$

Where $n = c - \frac{1}{2}c^2 + \frac{1}{3}c^3 - \text{\&c.} = (a-1) - \frac{1}{2}(a-1)^2 + \text{\&c.}$, and n_2, n_3, &c., also depend upon powers of c.

Hence we may assume that

$$a^x = 1 + nx + n_2x^2 + n_3x^3 + n_4x^4 + \text{\&c.};$$

$$\therefore\ a^{2x} = 1 + 2nx + 4n_2x^2 + 8n_3x^3 + 16n_4x^4 + \text{\&c.}$$

But $a^{2x}=a^x\times a^x=(a^x)^2$.

$$\therefore\ 1+2nx+4n_2x^2+8n_3x^3+16n_4x^4+\&c.$$
$$=1+2nx+(n^2+2n_2)x^2+(2n_3+2nn_2)x^3+$$
$$(2n_4+n_2{}^2+2nn_3)x^4+\&c.$$

Hence, equating the coefficients of the like powers of x,

$$4n_2=n^2+2n_2\,;\quad \therefore\ n_2=\tfrac{1}{2}n^2,$$

$$8n_3=2n_3+2n\,n_2\,;\ \therefore\ 6n_3=n^3\,;\ \therefore\ n_3=\frac{n^3}{2\cdot3}=\frac{n^3}{\underline{|3}}$$

$$\therefore\ 16n_4=2n_4+n_2{}^2+2n\,n_3\,;\ \therefore\ 14n_4=\tfrac{1}{4}n^4+\tfrac{1}{3}n^4$$

$$\therefore\ n_4=\frac{n^4}{2\cdot3\cdot4}\,;\ \&c.$$

and $\therefore\ a^x=1+nx+\dfrac{n^2x^2}{\underline{|2}}+\dfrac{n^3x^3}{\underline{|3}}+\dfrac{n^4x^4}{\underline{|4}}+\&c.$

Now take $nx=1$, then $x=\dfrac{1}{n}$, and

$$a^{\frac{1}{n}}=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\&c.$$

This series, denoted by the symbol e, is the base of the Napierian System of logarithms.

$$\because\ a^{\frac{1}{n}}=e,\ \therefore\ a=e^n,\ \therefore\ n=\log_e a\,;\ \text{hence}$$

$$a^x=1+(\log_e a)\,x+\frac{(\log_e a)^2x^2}{\underline{|2}}+\frac{(\log_e a)^3x^3}{\underline{|3}}+\&c.$$

This result is called the *Exponential Theorem.*

Cor. 1. Put e for a, then $\log_e a=\log_e e=1$; then

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\&c.$$

Cor. 2. Put 1 for x, then, $e=1+1+\dfrac{1}{\underline{|2}}+\dfrac{1}{\underline{|3}}+\dfrac{1}{\underline{|4}}+\&c.$

$$=2\cdot718281828\cdots$$

Cor. 3. Put mr for x, then $e^{mr}=1+mr+\dfrac{m^2r^2}{\underline{|2}}+\dfrac{m^3r^3}{\underline{|3}}+\&c.$

307. We will add here another proof of the Exponential Theorem.

By the Binomial Theorem

$$\left(1+\frac{1}{n}\right)^{nx}=1+nx\frac{1}{n}+\frac{nx(nx-1)}{\lfloor 2}\,\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{\lfloor 3}\,\frac{1}{n^3}+\&c.$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\&c.$$

Put 1 for x, then

$$\left(1+\frac{1}{n}\right)^{n}=1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\&c.$$

But $\left(1+\frac{1}{n}\right)^{nx}=\left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x}$; therefore

$$1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\&c.$$

$$=\left\{1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\&c.\right\}^{x}$$

Now suppose n to be infinite; then $\frac{1}{n}=0$,

$$\therefore\quad 1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\&c.=\left\{1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\&c.\right\}^{x}$$

$$=e^{x}.$$

Put rx for x, then

$$e^{rx}=1+rx+\frac{r^2x^2}{\lfloor 2}+\frac{r^3x^3}{\lfloor 3}+\&c.$$

Again let $a=e^{r}$, then $r=\log_e a$, and

$$a^{x}=1+\log_e a\,x+\frac{(\log_e a)^2x^2}{\lfloor 2}+\frac{(\log_e a)^3x^3}{\lfloor 3}+\&c.$$

308. To find the limit of the error committed by taking the first $(n+1)$ terms of the expansion of e.

The error

$$=\frac{1}{1\cdot2\cdot3\ldots(n+1)}+\frac{1}{1\cdot2\cdot3\cdot4\ldots(n+2)}+\frac{1}{1\cdot2\cdot3\cdot4\cdots(n+3)}+\&c.$$

$$=\frac{1}{\lfloor n}\left\{\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\cdots\right\}$$

$$<\frac{1}{\lfloor n}\left\{\frac{1}{n+1}+\frac{1}{(n+1)^2}+\&c.\right\}$$

$$<\frac{1}{\lfloor n}\times\frac{1}{n}.$$

Let $n=9$; error committed by taking 10 terms of the series

$$<\frac{1}{\lfloor 8\times9^2}\text{ or }\frac{1}{3265920}\text{ or }\cdot0000003.$$

LOGARITHMIC SERIES.

309. *To find the logarithm of a number in terms of the number itself.*

First find the expansion of $\log_e(1+x)$ in a series of ascending powers of x

$\because\ n=\log_e a$ (Art. 306)

$$=(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\&c.$$

$\therefore$ putting $1+x$ for a

$$\log_e(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\&c.$$

Likewise $\log_e(1-x)=-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4-\&c.$

$$\text{Now }\log_e\left(\frac{1+x}{1-x}\right)=\log_e(1+x)-\log_e(1-x).$$

$$\therefore\ \log_e\left(\frac{1+x}{1-x}\right)=2(x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\&c.)$$

Let $\dfrac{m-r}{m+r}=x$, then $\dfrac{1+x}{1-x}=\dfrac{m}{r}$.

$$\therefore\ \log_e\frac{m}{r}=2\left\{\frac{m-r}{m+r}+\tfrac{1}{3}\left(\frac{m-r}{m+r}\right)^3+\tfrac{1}{5}\left(\frac{m-r}{m+r}\right)^5+\&c.\right\}(A).$$

Let $r=1$, then

$$\log_e m = 2\left\{\frac{m-1}{m+1} + \frac{1}{3}\left(\frac{m-1}{m+1}\right)^3 + \frac{1}{5}\left(\frac{m-1}{m+1}\right)^5 + \&c.\right\} \qquad (B).$$

Also put $r+1$ for m in the identity (A), then

$$\log_e \frac{r+1}{r} \text{ or } \log_e(r+1) - \log_e r$$

$$= 2\left\{\frac{1}{2r+1} + \frac{1}{3(2r+1)^3} + \frac{1}{5(2r+1)^5} + \&c.\right\} \qquad (C).$$

From (B) we may obtain the value of $\log_e 2$; for m put 2,

then $\log_e 2 = 2\left(\frac{1}{3} + \frac{1}{3\times 3^3} + \frac{1}{5\times 3^5} + \ldots\ldots\right) = \cdot 693147179.$

Of any two consecutive numbers when we know the logarithm of one we may obtain the logarithm of the other from (C). Put 2 for r, then $\log_e 3 - \log_e 2$

$$= 2\left\{\frac{1}{5} + \frac{1}{3(5)^3} + \frac{1}{5(5)^5} + \ldots\ldots\right\} = \cdot 405465119$$

$$\therefore \log_e 3 = 1\cdot 098612288\ldots\ldots\ldots$$

310. To find the value of $\log_e 10$.

$$\log_e 10 - \log_e 9 = 2\left\{\frac{1}{19} + \frac{1}{3\times 19^3} + \frac{1}{5\times 19^5} + \&c.\right\} \text{ [Art. 309,}(C).$$

$$= \cdot 105360515$$

But $\log_e 9 = \log_e 3^2 = 2\log_e 3 = 2\times 1\cdot 098612288 = 2\cdot 197224577$;

$$\therefore \log_e 10 = \cdot 105360515 + 2\cdot 197224577 = 2\cdot 302585092.$$

311. The Napierian logarithms (from Napier the inventor of logarithms) are also called *natural* logarithms. The logarithms of numbers to the base 10 are used in practical calculations. The logarithm of a number to the base 10 is obtained by multiplying the Napierian logarithm by $\frac{1}{\log_e 10}$, that is, by $\frac{1}{2\cdot 302585092}$, or by $\cdot 434294481$ (Art. 301). This multiplier is called the *modulus* of the common system.

Examples worked out.

Ex. 1. To find the value of $e^{\frac{1}{10}}$ correct to 6 places of decimals.

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \&c.$$

$$\therefore\ e^{\frac{1}{10}} = 1 + \frac{1}{10} + \frac{1}{10^2 \times \lfloor 2} + \frac{1}{10^3 \lfloor 3} + \&c.$$

$$= 1 + \cdot 1 + \cdot 005 + \cdot 0001\dot{6} + \cdot 0000041\dot{6} + \cdot 00000008 + \&c.$$

$$= 1\cdot 105170.$$

Ex. 2. To find the error committed when the value of e is calculated to 6 terms of the series.

$$\text{The error} < \frac{1}{\lfloor 5 \times 5} \text{ or } \frac{1}{600} \text{ or } \cdot 001\dot{6}.$$

Ex. 3. In the expansion of $e^{\frac{1}{3}\left(1 - \frac{1}{x}\right)}$ find the term independent of x.

$$\because\ e^x = 1 + x + \frac{1}{\lfloor 2} x^2 + \frac{x^3}{\lfloor 3} + \cdots \&c.$$

$$e^{\frac{1}{3}\left(1 - \frac{1}{x}\right)} = 1 + \tfrac{1}{3}\left(1 - \frac{1}{x}\right) + \frac{1}{3^2 \lfloor 2}\left(1 - \frac{1}{x}\right)^2 + \frac{1}{3^3 \lfloor 3}\left(1 - \frac{1}{x}\right)^3,$$

$\therefore$ the term independent of x

$$= 1 + \tfrac{1}{3} + \frac{1}{3^2 \lfloor 2} + \frac{1}{3^3 \lfloor 3} + \&c.$$

Ex. 4. If x^3 and the higher powers of x be rejected, then shew that

$$e^{2x} = \frac{1 + x}{1 - x}.$$

$$e^{2x} = 1 + 2x + 2x^2 + \&c. = \frac{1 + x}{1 - x}.$$

Ex. 5. Find a series for $e + e^{-1}$

$$e = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \&c.$$

$$e^{-1}=1-1+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\&c.$$

$$\therefore e+e^{-1}=2\left\{1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\&c.\right\}.$$

Ex. 6. Prove that $\log_e 2=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\ldots\ldots+\&c.$

$$\because \log_e(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\ldots\ldots$$

Let $x=1$, then

$$\log_e 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\ldots\ldots\&c.$$

$$=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\frac{1}{7\cdot 8}+\&c.$$

EXAMPLES LXXIX.

1. Find the value of $e^{\frac{1}{1000}}$ correct to 8 places of decimals.

2. Find the value of $e^{-\frac{1}{10}}$ correct to 6 places of decimals.

3. Find the error committed when the value of e is calculated to 8 terms of the series.

4. Find the error committed when the value of $e^{\frac{1}{2}}$ is calculated to 6 terms of the series.

5. Find the error committed when the value of $e^{\frac{1}{10}}$ is calculated to 8 terms of the series.

6. Find the number of terms of the series of $e^{\frac{1}{10}}$ that must be taken so as to calculate correctly to 6 places of decimals.

7. Prove that $r=1+\log_e r+\frac{(\log_e r)^2}{\lfloor 2}+\frac{(\log_e r)^3}{\lfloor 3}+\&c.$

8. Prove that $e-e^{-1}=2\left\{1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\frac{1}{\lfloor 7}+\&c.\right\}.$

9. Obtain a series for $\frac{1}{2}(e+e^{-1}-2)$.

10. Prove that $\log_e 2=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\frac{1}{7\cdot 8}+\&c.$

11. Prove that $\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \&c....\right)$.

12. Supposing that the value of $\log_e 2$ is known, what values must be substituted for n and x in the formula

$$\log_e(n+x) = \log_e n + 2\left\{\frac{x}{2n+x} + \frac{x^3}{3(2n+x)^3} + \frac{x^5}{5(2n+x)^5} + \&c....\right\}$$

in order to obtain the Napierian logarithm of 12?

(In the following 5 examples give the answers correct to 6 places of decimals only.)

13. $\text{Log}_e 3 = 1{\cdot}098612288$, find the value of $\log_e 4$.

14. $\text{Log}_e 10 = 2{\cdot}302585092$, find the value of $\log_e 3$.

15. $\text{Log}_e 2 = {\cdot}693147179$, $\log_e 10 = 2{\cdot}302585092$, find $\log_e 25$.

16. $\text{Log}_e 2 = {\cdot}693147179$, find $\log_e 12$.

17. $\text{Log}_e 5 = 1{\cdot}6094379$, find $\log_e 36$.

18. Expand $\left(1+\frac{1}{n}\right)^{nx}$, and show that when n is indefinitely great the expansion is the same as the known expansion for e^x.

19. In the expansion of $(e^x - 1)^6$, show that the coefficient of x^8 vanishes.

20. Find the Napierian logarithm of $\frac{1001}{999}$ correct to 7 places of decimals.

21. Prove that $\frac{1}{1{\cdot}2{\cdot}3} + \frac{1}{3{\cdot}4{\cdot}5} + \frac{1}{5{\cdot}6{\cdot}7} + \&c. = \log_e 2 - \frac{1}{2}$.

22. Prove that $\frac{1}{1{\cdot}2{\cdot}3{\cdot}4} + \frac{1}{3{\cdot}4{\cdot}5{\cdot}6} + \frac{1}{5{\cdot}6{\cdot}7{\cdot}8} + \&c. = \frac{2}{3}\log_e 2 - \frac{5}{12}$.

23. From the expansion of $e^{-x}(e^x - 1)^{n+1}$ prove that

$$n^n - (n+1)(n-1)^n + \frac{(n+1)n}{\lfloor 2}(n-2)^n - ...\&c. \text{ to } n \text{ terms} = 1.$$

24. From the expansion of $e^{-x}(e^x - 1)^n$, prove that

$$(n-1)^n - n(n-2)^n + \frac{n(n-1)}{\lfloor 2}(n-3)^n - \&c. \text{ to } (n-1)\text{terms} = \lfloor n-1.$$

Interest and Discount.

312. *Interest* is the consideration paid for the use of money. The *rate* of interest is the consideration paid for the use of a certain sum for a certain time, as of 1 Rupee for one year.

The sum lent is called the *Principal.*

The amount is the whole sum due at the end of any time, *Interest* and *Principal* together.

313. Interest is of two kinds, *simple* and *compound.*

When the interest of the *principal* alone is taken, it is called *simple* interest.

If the interest, as soon as it becomes due, be added to the principal, and interest be charged upon the whole, it is called *compound* interest.

314. *To find the amount of a given sum, in any time at simple interest.*

Let P be the principal, n, the number of years for which the interest is to be calculated, r the interest of one Rupee for one year, m, the amount.

Then since the interest of a given sum, at a given rate must be proportional to the time.

$$1 \text{ (year)} : n \text{ (years)} :: r : nr.$$

$$\therefore\ nr = \text{the interest of 1 Rupee for } n \text{ years};$$

and the interest of P rupees must be P times as great or nrP.

$$\therefore \text{ the amount or } M = P + Pnr.$$

315. In the above simple equation, any three of the quantities P, n, r, M being given, the fourth may be found;

$$\text{thus } P = \frac{M}{1+nr},\ r = \frac{M-P}{Pn},\ n = \frac{M-P}{Pr}.$$

Ex. What principal, in eight months, will amount to Rs 4620, allowing interest at the rate of 4 per cent. per annum?

In this case, $M = 4620$, $n = 8$ months $= \frac{2}{3}$ year.

$$r = \tfrac{4}{100} = \tfrac{1}{25}$$

$$\text{Hence } P = \frac{M}{1+nr} = \frac{4620}{1+\frac{2}{3}\times\frac{1}{25}} = \text{Rs } 4500.$$

316. *To find the amount of a given sum in any time at compound interest.*

Let $R=1$ Rupee together with its interest for 1 year.

Then at the end of the first year, R becomes the principal, or the sum due.

$\therefore\ 1 : R :: R : R^2$ $\therefore\ R^2=$ the amount in 2 years.

$1 : R :: R^2 : R^3$ $\therefore\ R^3=$ „ „ „ 3 years. &c.

In the same manner $R^n=$ „ „ „ n years.

If P be the principal, the amount must be P times as great, or $PR^n=M$.

$$\text{Hence } P=\frac{M}{R^n},\ R=\sqrt[n]{\frac{M}{P}},\ n=\frac{\log M-\log P}{\log R}.$$

Ex. What sum must be paid down to receive Rs 9261 at the end of three years, allowing 5 per cent. per annum compound interest.

In this case, the interest of 1 Rupee per annum $=\frac{5}{100}=\cdot05$,

$$\therefore\ R=1+\cdot05=1\cdot05.$$

$$n=3,\ M=9261.$$

$$\therefore\ P=\frac{9261}{(1\cdot05)^3}=\frac{9261\times(20)^3}{(21)^3}=\text{Rs }8000.$$

317. *To find the amount when interest is reckoned* half yearly, *when* quarterly, *and when* p *times a year.*

If $r=$ interest of one Rupee for one year, and n the number of years, then, if interest is paid *half-yearly*, $1+\frac{1}{2}r=$ amount of 1 Rupee at the end of the first payment, and $2n$ is the whole number of payments,

$$\therefore\ M=P(1+\tfrac{1}{2}r)^{2n}.$$

Similarly, when interest is paid *quarterly*,

$$M=P(1+\tfrac{1}{4}r)^{4n}.$$

When interest is paid p times a year, the amount of 1 Rupee at the end of the first payment is $1+\frac{1}{p}r$ and $pn=$ number of years.

$$\therefore\ M=P\left(1+\frac{r}{p}\right)^{pn}.$$

318. The *Present Worth* of an amount due at the end of a given time is that sum which together with its interest for the time is equal to the given amount.

Discount is the abatement made for the payment of money before it is due.

Hence discount is the difference between the amount due at the end of the given time and the present worth.

319. *To find the present worth of a sum due at the end of a given time and the discount.*

Let P be the present worth, n the number of years, M the sum due at the end of n years, D the discount, r the interest of one rupee for one year and R the amount of one rupee in one year.

Then it is clear that if P be put to interest, its amount ought in fairness to be equal to M.

At simple interest,

$P+Pnr=M$; (Art. 314)

$$\therefore\ P=\frac{M}{1+nr} \text{ and } D=M-P=\frac{Mnr}{1+nr}.$$

At compound interest

$M=PR^n$. (Art. 316).

$$\therefore\ P=\frac{M}{R^n},\ D=M-P=\frac{M(R^n-1)}{R^n}.$$

320. In practice, the bankers and merchants invariably deduct the interest on the whole sum from the present time till it becomes due instead of the true discount which is the interest of the present worth. Thus they deduct Mnr instead of $\frac{Mnr}{1+nr}$.

Ex. To find the present worth and the discount of Rs 828 due at the end of three years at 5 per cent. simple interest; also find the banker's discount.

In this case $M=828$, $n=3$, $r=\cdot05$.

$$\therefore\ P=\frac{M}{1+nr}=\frac{828}{1+\cdot15}=\text{Rs }720.$$

Discount $=828-720=$ Rs 108.

The banker's discount $=828\times\cdot15=$ Rs $124\cdot2$.

EXAMPLES LXXX.

1. What sum in 3 years at 5 per cent. per annum simple interest will amount to Rs 5175?

2. What sum of money put out to compound interest for 3 years at 4 per cent. per annum, will amount to Rs 17576?

3. Find the discount of Rs 1035 due 3 years hence at 5 per cent. per annum simple interest.

4. Find the discount of Rs 4630 8 as. due 3 years hence at 5 per cent. per annum compound interest.

5. What sum of money put out to compound interest for 18 years at 4 per cent. per annum, will amount to Rs 10000?

Having given $\log 104 = 2{\cdot}0170333$ and
$\log 493629 = 5{\cdot}6934006$.

6. What would be the amount of Rs 2000 placed out to compound interest for 7 years at 4 per cent. per annum?

Given $\log 10{\cdot}4 = 1{\cdot}0170333$ and $\log 263186 = 5{\cdot}4202631$.

7. At what rate of interest must Rs 4000 be placed out, that it may amount to Rs 5693 4 as. in 9 years at compound interest?

Given $\log 10{\cdot}4 = 1{\cdot}0170333$, $\log 4 = {\cdot}6020600$ and
$\log 569{\cdot}325 = 2{\cdot}7553600$.

8. In how many years will Rs 1000 amount to Rs 1600 at 6 per cent. per annum compound interest?

Given $\log 2 = {\cdot}3010300$, $\log 106 = 2{\cdot}0253059$.

9. In what time will a sum of money *double* and *treble* itself at 5 per cent. per annum compound interest?

Given $\log 2 = {\cdot}3010300$, $\log 3 = {\cdot}4771213$, $\log 105 = 2{\cdot}0211893$.

10. Supposing the interest to be paid *half-yearly*; what would be the amount of Rs 100000 placed out for ten years at 10 per cent. per annum compound interest?

Given $\log 2 = {\cdot}3010300$, $\log 26532975 = 7{\cdot}4237860$.

11. In what time will £1000 amount to £131501 at 5 per cent. per annum compound interest?

Given $\log 131{\cdot}501 = 2{\cdot}1189300$, $\log 105 = 2{\cdot}0211893$.

12. In what time will £500 amount to £742 5s. at 5 per cent. per annum, compound interest, the interest being paid *half-yearly*, having given,

$\log 14845 = 4{\cdot}1715802$, $\log 1{\cdot}025 = {\cdot}0107239$.

Annuities.

321. *To find the amount of an annuity, or pension left unpaid any number of years, allowing simple interest upon each sum or pension from the time it becomes due.*

Let A be the annuity, n the number of years, r the interest of one Rupee for one year, M the amount.

At the end of the first year, A becomes due, and at the end of the second year, the interest of the first annuity is rA; at the end of this year, the principal becomes $2A$, therefore the interest due at the end of the third year is $2rA$; in the same manner, the interest due at the end of the fourth year is $3rA$; &c.; hence the whole interest is $rA+2rA+3rA+\ldots+(n-1)rA=\frac{1}{2}n(n-1)rA$; and the sum of the annuities is nA; therefore the whole amount

$$M=nA+\tfrac{1}{2}n(n-1)rA.$$

322. *Required the present value of an annuity to continue a certain number of years, allowing simple interest for the money.*

Let P be the present value; then if P, and the annuity, at the same rate of interest, amount to the same sum, they are upon the whole of equal value. The amount of P, in n years, is $P+nrP$ (Art. 314); and the amount of the annuity in the same time is $nA+\frac{1}{2}n(n-1)rA$;

$$\text{therefore } P+nrP=nA+\tfrac{1}{2}n(n-1)rA,$$

$$\text{and } P=\frac{nA+\frac{1}{2}n(n-1)rA}{1+nr}.$$

In this equation any three of the four quantities P, A, n, r being given, the other may be found.

323. COR. Let n be infinite, then $P=\frac{1}{2}nA$ an infinite quantity, therefore for a finite annuity to continue for ever, an infinite sum ought, according to this calculation, to be paid: a conclusion which shews the necessity of estimating the value of an annuity upon different principles.

324. *To find the amount of an annuity in any number of years at compound interest.*

Let A be the annuity or sum due at the end of the first year; r the interest of 1 Rupee in one year; let $R=1+r=$ amount of one rupee in one year, then $1 : R :: A : RA$.

$\therefore$ $RA=$ its amount at the end of the 2nd year.

$\therefore$ $A+RA$ is the sum due at the end of the 2nd year.

In the same manner, $1 : R :: (1+R)A : (R+R^2)A$.

$\therefore$ $(R+R^2)A =$ amount of the two payments at the third year, and $(1+R+R^2)A =$ whole sum due at the end of the third year.

Similarly, $(1+R+R^2+\cdots+R^{n-1})A$ is the sum due at the end of n years, that is

$$M=\frac{R^n-1}{R-1}A.$$

COR. 1. In this equation, any three of the quantities being given, the fourth may be found.

$$A=\frac{R-1}{R^n-1}M,\ n=\frac{\log\left\{1+\frac{M}{A}(R-1)\right\}}{\log R}.$$

COR. 2. If interest be payable p times a year, and $\frac{r}{p}$ be each payment per 1 Rupee, then $\left(1+\frac{r}{p}\right)^p=$ amount of 1 Rupee in one year,

$$\text{then } M=\frac{\left(1+\frac{r}{p}\right)^{np}-1}{\left(1+\frac{r}{p}\right)^{p}-1}A.$$

COR. 3. If the annuity be payable x times a year, each payment $=\frac{A}{x}$, and m be the annual rate of interest, then $\sqrt[x]{(1+m)}=$ amount of 1 Rupee in x^{th} part of a year, $\therefore$ amount in n years

$$=\frac{A}{x}\cdot\frac{\{\sqrt[x]{(1+m)}\}^{nx}-1}{\sqrt[x]{(1+m)}-1}=\frac{A}{x}\cdot\frac{(1+m)^n-1}{\sqrt[x]{(1+m)}-1}.$$

If the interest be also payable p times a year, then

$$M=\frac{A}{x}\cdot\frac{\left(1+\frac{m}{p}\right)^{np}-1}{\left(1+\frac{m}{p}\right)^{\frac{p}{x}}-1}.$$

325. *To find the present value of an annuity to be paid for n years*, allowing compound interest.

Let P be the present value, A the annuity; then since PR^n is the amount of P in n years, and $\frac{R^n-1}{R-1}A$, the amount of A in the same time; by the question,

$$PR^n=\frac{R^n-1}{R-1}A, \text{ and } P=\frac{1-\frac{1}{R^n}}{R-1}A.$$

COR. 1. Any three of the quantities P, A, R, n being given the fourth may be found.

COR. 2. If the number of years be infinite, R^n is infinite and $\frac{1}{R^n}$ vanishes; therefore $P=\frac{A}{R-1}=\frac{A}{r}$.

Ex. If the annual rent of a freehold estate be Rupee 1 what is its value, allowing 5 per cent. per annum compound interest?

In this case $A=1$, $R-1$ or $r=\cdot05$; therefore the present value $=\frac{1}{\cdot05}=$ Rs 20, or 20 years' purchase.

326. The *Present Value* of an *Annuity*, to commence at the expiration of p years and to continue q years, is the difference between its present value for $p+q$ years, and its present value for p years. Therefore the present value

$$=\frac{A-AR^{-(p+q)}}{R-1}-\frac{A-AR^{-p}}{R-1}$$

$$=\frac{AR^{-p}-AR^{-(p+q)}}{R-1}=\frac{A}{R-1}\left(R^{-p}-R^{-p-q}\right).$$

Ex. What is the present value of an annuity of 1 Rupee, for 14 years, to commence at the expiration of 7 years, allowing 5 per cent. per annum compound interest.

The present value for 21 years $=\frac{(1\cdot05)^{21}-1}{(1\cdot05)^{21}\times\cdot05}$

The present value for 7 years $=\frac{(1\cdot05)^{7}-1}{(1\cdot05)^{7}\times\cdot05}$

$\therefore$ the required present value $=\frac{1}{\cdot05}\left\{\frac{1}{(1\cdot05)^7}-\frac{1}{(1\cdot05)^{21}}\right\}$

$=20(x-y)$ suppose.

$\log x = -\ \ 7 \log (1{\cdot}05) = \bar{1}{\cdot}8516749 = \log {\cdot}710681.$
$\log y = -21 \log (1{\cdot}05) = \bar{1}{\cdot}5550247 = \log {\cdot}358942.$

$$\therefore\ x = {\cdot}710681 \text{ and } y = {\cdot}358942.$$

$$x - y = {\cdot}351739.$$

Present value $= 20 \times {\cdot}351739 =$ ~~Rs~~ $7{\cdot}035$ nearly.

Cor. If the annuity is to continue for ever, q is infinite, and the present value $= \dfrac{AR^{-p}}{R-1}$.

Scholium.

327. The method of determining the present value of an annuity at simple interest, given in Art. 322, has been decried by several eminent Arithmeticians, and in its stead, a solution of the question has been proposed upon the following principle; "If the present value of each payment be determined separately, the sum of these values must be the value of the whole annuity."

Let x be the value or price paid down for the annuity, a the yearly payment, n the number of years for which it is to be paid, r the interest of 1 Rupee for one year. The present value of the first payment is $\dfrac{a}{1+r}$, the present value of the second payment, or of a Rupees to be paid at the end of two years, is $\dfrac{a}{1+2r}$, and so on; therefore

$$x = \frac{a}{1+r} + \frac{a}{1+2r} + \frac{a}{1+3r} + \dots + \frac{a}{1+nr}.$$

328. These different conclusions arise from a circumstance which the opponents seem not to have attended to. According to the former solution, no part of the interest of the price paid down is employed in paying the annuity, till the principal is *exhausted*.

Let the annuity be always paid out of the principal x as long as it lasts, and afterwards out of the interest which has accrued; then x, $x-a$, $x-2a$, $x-3a$, &c., are the sums in hand, during the first, second, third and fourth &c., years, the interest arising from which, rx, $rx-ra$, $rx-2ra$, $rx-3ra$, &c.,

that is, the whole interest is $nrx-\{1+2+3+\ldots\ldots+(n-1)\}ra$; or $nrx-\frac{1}{2}n(n-1)ra$, which, together with the principal x, is equal to the sum of all the annuities; therefore $(1+nr)x-\frac{1}{2}n(n-1)ra=na$, and

$$x=\frac{na+\frac{1}{2}n(n-1)ra}{1+nr} \qquad \text{(Art. 322).}$$

According to the other calculation, part of the interest, as it arises, is employed in paying the annuity, but not the whole. Thus, the first payment is made by a part of the principal, and the interest of that part, which together amount to the annuity; and the other payments are made in the same manner; this is, in effect, allowing interest upon that part of the whole interest which is incorporated with the principal. According to either calculation, the seller has the advantage, since the whole, or part of the interest will remain at his disposal till the last annuity is paid off.

If the whole interest, as it arises, be incorporated with the principal, and employed in paying the annuity, compound interest is, in effect, allowed upon the whole. Let x be the price paid for the annuity, n the number of years for which it is granted, and $R=1$ Rupee together with its interest for one year. Then x in one year amounts to Rx, out of which the annuity being paid, $Rx-a$ is the sum in hand at the end of the first year; R^2x-Ra is the amount of this sum at the end of the second year, therefore $R^2x-Ra-a$ is the sum in hand at the end of the second year; in the same manner, $R^nx-R^{n-1}a-R^{n-2}a\ldots\ldots-a$ is the sum left, after paying the last annuity, which ought to be nothing; therefore

$$R^nx=R^{n-1}a+R^{n-2}a\ldots\ldots+a=\frac{R^na-a}{R-1},$$

$$\text{and } x=\frac{R^na-a}{R^n(R-1)}. \quad \text{Art. 325.}$$

Examples worked out.

Ex. 1. To find the amount of an annuity of Rs 100 left unpaid for 10 years, allowing 5 per cent. per annum simple interest upon each sum from the time it becomes due.

In this case $n=10$, $A=100$, $r=\cdot05$.

$$\begin{aligned}M&=nA+\tfrac{1}{2}n(n-1)rA\\&=1000+\tfrac{1}{2}\times10\times9\times\cdot05\times100\\&=1000+225=\text{Rs }1225.\end{aligned}$$

Ex. 2. To find the present value of an annuity of ₨120 to continue for 20 years, allowing 4 per cent. per annum simple interest for the money.

In this case $A=120$, $n=20$, $r=\cdot04$.

$$P=\frac{nA+\frac{1}{2}n(n-1)rA}{1+nr}=\frac{120\times20+\frac{1}{2}\times20\times19\times\cdot04\times120}{1+20\times\cdot04}$$

$$=\frac{3312}{1\cdot8}=\text{₨}1840.$$

Ex. 3. The present value, of an annuity of ₨200 to continue for n years allowing 5 per cent. per annum simple interest for the money, is ₨1912 8as. Find n.

$$1912\cdot5=200\frac{n+\frac{1}{2}n(n-1)\times\cdot05}{1+\cdot05n}.$$

$$n^2+19\cdot875n-382\cdot5=0.$$

Hence $n=12$.

Ex. 4. To find the amount of an annuity of ₨100 left unpaid for 10 years, allowing 5 per cent. per annum compound interest upon each sum from the time it becomes due.

Given $\log 1\cdot05=\cdot0211893$ and $\log 1\cdot6289=\cdot211893$.

Here $A=100$, $n=10$, $r=\cdot05$, $R=r+1=1\cdot05$.

$$M=\frac{R^n-1}{R-1}A=\frac{(1\cdot05)^{10}-1}{1\cdot05-1}A=\frac{(1\cdot05)^{10}-1}{\cdot05}\times100.$$

Let $x=(1\cdot05)^{10}$.

$$\text{Log}\,x=10\log 1\cdot05=10\times\cdot0211893$$
$$=\cdot211893=\log 1\cdot6289;\quad\therefore\quad(1\cdot05)^{10}=1\cdot6289.$$

$$\therefore\quad M=\frac{1\cdot6289-1}{\cdot05}\times100=\text{₨}1257\cdot8\text{ nearly.}$$

Ex. 5. To find the present value of an annuity of ₨100 to be paid for 20 years, allowing 5 per cent. per annum compound interest.

$A=100$, $n=20$, $R=1\cdot05$.

$$P=\frac{1-R^{-n}}{R-1}A=\frac{1-(1\cdot05)^{-n}}{1\cdot05-1}A=\frac{1-(1\cdot05)^{-20}}{\cdot05}\times100.$$

Let $(1\cdot05)^{-20}=x$.

$$\text{Log } x=-20\times\cdot0211893$$
$$=\bar{1}\cdot576214=\log\cdot37689,$$

$\therefore\ (1\cdot05)^{-20}=\cdot37689$.

$\therefore\ P=(1-\cdot37689)\times2000$ Rs $=$ Rs $1246\cdot22$ nearly.

Ex. 6. The annual rent of a free-hold estate is, Rs 1000; what is its value, allowing 4 per cent. per annum compound interest?

$A=1000,\ R-1=\cdot04$.

Present value $=\dfrac{A}{R-1}=\dfrac{1000}{\cdot04}=$ Rs 25000 or 25 years' purchase.

Ex. 7. An annuity of Rs 200 a year, for 20 years, is to commence at the expiration of 10 years. Find its present value, allowing 4 per cent. per annum compound interest.

Present value $=\dfrac{1}{\cdot04}\left\{\dfrac{1}{(1\cdot04)^{10}}-\dfrac{1}{(1\cdot04)^{30}}\right\}\times200$.

[See Ex. Art. 326.]

$$=5000(x-y) \text{ suppose.}$$

$\text{Log } x=-10\log(1\cdot04)=\bar{1}\cdot829667=\log\cdot675565$.

$\text{Log } y=-30\log(1\cdot04)=\bar{1}\cdot488999=\log\cdot308311$.

$\therefore$ present value $=5000(\cdot675565-\cdot308311)$
$=$ Rs $1836\cdot27$ nearly.

EXAMPLES LXXXI.

1. Find the amount of an annuity of Rs 600 left unpaid for 16 years, allowing 4 per cent. per annum simple interest upon each sum from the time it becomes due.

2. Find the present value of an annuity of Rs 4000 to continue for 25 years, allowing 5 per cent. per annum simple interest for the money.

3. Find the present value in the above example if compound interest is allowed. Given $\log 1\cdot05=\cdot0211893$, $\log 2\cdot95362=\cdot4702675$.

In the following examples the interest is to be supposed *compound* unless otherwise mentioned.

4. A person deposits in a bank Rs 36 on the 1st of January of each year; the banker invests the money at 4 per cent. for n years when the whole amounts to Rs 1000. Find n; having given $\log 10\cdot4 = 1\cdot0170333$, $\log 19 = 1\cdot278736$, $\log 9 = \cdot9542425$.

5. A person borrows Rs 10000 on condition to repay it with interest at 5 per cent., by annual instalments of Rs 1000 each. In how many years will the debt be paid off? Given $\log 2 = \cdot3010300$, $\log 1\cdot05 = \cdot0211893$.

6. What is the value of a free-hold estate producing a rent of Rs 4000, money making 4 per cent. per annum?

7. A person bought a free-hold estate at Rs 40000; what should be the nett annual income from the estate if money makes 5 per cent. per annum?

8. Two persons A and B purchased a free-hold estate at Rs 50,000; A is to enjoy an equal share for 10 years only after which B shall remain sole proprietor. What should A contribute towards the purchase money if money makes 4 per cent. per annum? Given $\log 1\cdot04 = \cdot0170333$, $\log 6\cdot75564 = \cdot829667$.

9. What is the present value of an annuity of Rs 700 to commence six years hence, and then to continue for 21 years, allowing interest at 5 per cent?

Given $\log 1\cdot05 = \cdot0211893$, $\log 746215 = 5\cdot8728642$ and $\log 26\cdot7849 = 1\cdot4278889$.

10. A person invests annually the sum of Rs 100 at 5 per cent. per annum for 40 successive years. What will he receive at the end of 40 years?

Given $\log 1\cdot05$ and $\log 7\cdot39199 = \cdot8687613$.

11. What sum of money must a person invest at 4 per cents. so as just to produce an income of Rs 1200 a year for 20 years, the principal being exhausted at the end of the time, and the income being received half-yearly?

Given $\log 1\cdot04 = \cdot0170333$

$\log 4\cdot56388 = \cdot6593340$

$\log 1\cdot019803 = \cdot0085166$.

12. What is the present value of an annuity of Rs 1600 for 30 years, payable quarterly, when the interest at the rate of 4 per cent. per annum is payable half-yearly?

Given $\log 102 = \cdot 0086002$ and $\log 1\cdot 0095 = \cdot 0043001$,
$\log 3\cdot 04781 = \cdot 483988$.

13. A person deposited with a banker Rs 75000 at 4 per cent. interest per annum but took from the banker Rs 5000 at the end of every year. In how many years will his funds be exhausted?

Given $\log 1\cdot 04 = \cdot 0170333$ and $\log 2 = \cdot 3010300$.

14. Out of a capital of Rs 100000 invested at 4 per cent interest payable quarterly, a person spends every half-year Rs 3000. Find the balance of his capital after 10 years, also find the time in which his capital will be reduced to Rs 25000. Having given $\log 1\cdot 01 = \cdot 0043214$, $\log 11 = 1\cdot 0413927$, $\log 1\cdot 48887 = \cdot 172856$, and $\log 27\cdot 75 = 1\cdot 4432630$.

15. An annuity of Rs 1000 is to commence at the expiration of 10 years, and to continue for 30 years, find the equivalent annuity to commence immediately and to continue 30 years, money making 5 per cent. per annum. Given $\log 1\cdot 05 = \cdot 0211893$ and $\log 6\cdot 13913 = \cdot 788107$.

Indeterminate Equations of the first degree.

329. When there are more unknown quantities than independent equations the number of corresponding values which those quantities admit, is indefinite (Art. 118). This number may be lessened, by rejecting all the values which are not integers; it may be farther lessened, by rejecting all the negative values; &c. By restrictions of this kind, the number of answers may be confined within definite limits.

330. *To solve a simple equation involving two unknown quantities.*

Ex. To find the positive integral values of x and y in the equation $3x+4y=23$.

Divide the equation by 3 the less of the two coefficients,

$$\text{then, } x+y+\frac{y}{3}=7+\tfrac{2}{3}\,; \quad \therefore\ x+\frac{y-2}{3}=7-y.$$

Now, if x be a whole number, $\frac{y-2}{3}$ must be a whole number or 0.

$$\text{Assume } \frac{y-2}{3}=n \text{ a whole number ;}$$

$$\text{then } y=3n+2 \text{ and } x=7-3n-2-n=5-4n.$$

Substitute 0 or any whole number for n, then

$$\left.\begin{array}{r} x=5 \text{ and } y=2 \text{ when } n=0, \\ x=1 \text{ and } y=5 \text{ when } n=1, \end{array}\right\} \text{ The only positive integral values.}$$
$$x=-3 \text{ and } y=8 \text{ when } n=2,$$

&c. &c. &c.

331. From the above we obtain the following rule :—

RULE. *If a simple equation express the relation of two unknown quantities and their corresponding integral values be required; divide the whole equation by the coefficient which is the less of the two, and suppose that part of the quotient, which is in a fractional form, equal to some whole number; thus a new simple equation is obtained, with which we may proceed as before; let the operation be repeated till the coefficient of one of the unknown quantities is unity, and the coefficient of the other a whole number; then an integral value of the former may be obtained substituting 0, or any whole number for the other; and from the preceding equations, integral values of the quantities proposed may be found; if now the negative values be rejected we will obtain the positive integral values which satisfy the given equation.*

332. *The equation* $\mathrm{ax} \pm \mathrm{by} = \mathrm{r}$ *will have no integral solution if* a *and* b *have a common divisor, not common also to* r.

Let $a=mc$ and $b=nc$, while r does not contain c, then

$$mcx \pm ncy = r, \quad \therefore \; mx+ny=\frac{r}{c}=\text{a fraction.}$$

But if the values of x and y be integers, $\frac{r}{c}$ must also be an integer, which is impossible.

Wherefore, *the equation* $\mathrm{ax}+\mathrm{by}=\mathrm{r}$, *will have no &c., &c.*

333. *Given one solution of the equation* $\mathrm{ax}+\mathrm{by}=\mathrm{c}$, to obtain *the general solution.*

Let $x=\alpha$, $y=\beta$ be *one* solution ; $\quad \therefore \; a\alpha+b\beta=c.$

By subtraction, $a(x-\alpha)+b(y-\beta)=0$.

$$\therefore\ x-\alpha=\frac{(\beta-y)b}{a}=bk,$$ where k is any integer positive or negative.

$$\text{and } y-\beta=-\frac{a(x-\alpha)}{b}=-ak.$$

$$\therefore\ x=\alpha+bk \text{ and } y=\beta-ak.$$

Cor. Similarly we obtain the general solution of th equation $ax-by=c$.

Examples worked out.

Ex. 1. To find a number which being divided by 3 and 4 the remainders are 2 and 3 respectively.

Let x be the number, then

$$\frac{x-2}{3}=p, \text{ a whole number}; \quad \therefore\ x=3p+2.$$

$$\text{Also } \frac{x-3}{4} \text{ or } \frac{3p-1}{4}=q, \text{ a whole number};$$

$$\text{that is, } 3p-1=4q. \qquad \therefore\ p=q+\frac{q+1}{3};$$

$$\text{let } \frac{q+1}{3}=r, \text{ or } q=3r-1;$$

$$\therefore\ p=4r-1 \text{ and } x=3p+2=12r-1.$$

Let $r=1, 2, 3$, &c. successively; $\therefore\ x=11, 23, 35, 47$, &c.

Ex. 2. A person distributed Rs 48 among some men and women, giving Rs 5 to each man and Rs 7 to each woman, find the number of each.

Let x = number of men and y = number of women.

$$\therefore\ 5x+7y=48$$

$$\therefore\ x+y+\frac{2y}{5}=9+\frac{3}{5}; \qquad \therefore\ x=9-y-\frac{2y-3}{5}.$$

$$\therefore\ \frac{2y-3}{5}=r \text{ suppose, a whole number.}$$

$$\therefore\ 2y=5r+3\ ;\quad \therefore\ y=2r+1+\frac{r+1}{2}.$$

$$\therefore\ \frac{r+1}{2}=t,\ \text{a whole number.}\quad \therefore\ r=2t-1\ ;$$

$$\therefore\ y=4t-2+1+t=5t-1.$$

$$\therefore\ x=9-5t+1-2t+1=11-7t.$$

Let $t=1, 2, 3,$ &c.,

$\therefore\ x=4$ and $y=4$, the *only* solution.

334. If the simple equation contain more than two unknown quantities, their corresponding integral values may be found in the same manner as in Art. 2.

EXAMPLES LXXXII.

1. Find the positive integral solutions of

(1) $2x+3y=23.$	(2) $3x+4y=24.$	(3) $3x+5y=22.$
(4) $4x+5y=31.$	(5) $7x-3y=22.$	(6) $9x-5y=4.$
(7) $9x-11y=3.$	(8) $11x-13y=7.$	(9) $14x-13y=29.$
(10) $50x-7y=36.$	(11) $21x-10y=95.$	(12) $73x-9y=126.$

2. In how many ways could Rs 2 be made up of half-rupees and quarter-rupees.

3. In how many ways may mangoes and cocoa-nuts be bought with 8 annas, paying 3 ps. for each mangoe and 2ps. for each nut?

4. In how many ways may Rs 200 be paid to some women and men, paying each man Rs 17 and each woman Rs 10?

5. I owe a friend Rs 105, but have sixteen-rupee Gold Mohurs to pay with; my friend has only five-rupee notes, how are we most easily to settle the account?

6. Find a number which being divided by 4 and 5, the remainders are 3 and 4 respectively.

7. Find the least whole number which, when divided by 2 and 5, has 1 and 3 as respective remainders.

8. The entertainment of a certain number of men and boys did cost Rs 33; the cost for each man was Rs 3 8 as. and for each boy R 1 4 as.; find the number of men and boys.

9. Find a number less than 200, which is a multiple of 5, and upon being divided by 2, 3, 4, always leaves 1 for a remainder.

10. How many ways are there of paying Rs 75 with twenty-rupee notes and five-rupee notes ?

11. Divide 16 into three such parts that if the first be multiplied by 20, the second by 3 and the third by 2, the sum of these products may be 129.

12. Thirty-six persons, men, women, and children, spent Rs150, whereof each man paid Rs10, each woman Rs5 and each child Rs3. How many were there of each ?

13. A person bought horses and bullocks for Rs330 ; the horses cost Rs50 each and the bullocks Rs30 each. How many of each did he buy ?

14. Find two numbers, such that if the first be multiplied by 5 and the second by 4, the first product shall exceed the second product by 33, the sum of the two numbers not exceeding 20.

15. A person counting a basket of mangoes, which he knows are between 70 and 80, finds that when he counts them 4 at a time there are 3 over and when 5 at a time, there are 4 over ; how many were there in all ?

16. Divide 100 into two such parts that one of them may be divisible by 5 and the other by 11.

17. There are two measures for measuring oil, one a three-seer measure and another a five-seer measure. In how many ways may 1 maund of oil be measured ?

18. Find the least whole number which, being divided by 9, 11, 13 shall leave remainders 2, 3 and 6 respectively.

19. A person bought some goats, sheep and fowls for Rs24, paying Rs3 for each goat, Rs4 for each sheep and R 1 for each fowl, the number of fowls being not more than 2. Find the number of each.

20. A person buys 20 animals of 3 different kinds for Rs2000. For the first he gives Rs200 each, for the second Rs100 and for the third Rs40. How many of each did he buy ?

Miscellaneous Examples.

1. Solve the following equations :—

(1) $7x^2-3x=100$.

(2) $x^2+\frac{5}{2}x-2\sqrt{(2x^2+5x+3)}=4\frac{1}{2}$.

(3) $\left.\begin{array}{l} x^2+2y^2=22 \\ 2xy+y^2=21. \end{array}\right\}$

2. In each of three battles 36 officers, and 10 per cent. of the men engaged, are killed. At the end of the second battle the percentage of officers to men is $\frac{2}{3}$ of what it was at its commencement; and the number of men at the end of the third is the square of the number of officers at its commencement. How many officers and men were engaged in the first battle?

3. Find an Arithmetical Progression such that the sum of n terms shall be equal to n^2, the first term being unity.

4. Solve the following equations :—

(1) $\frac{a}{\sqrt{x}}+\frac{b}{y^2}=m$, $\frac{b}{\sqrt{x}}+\frac{a}{y^2}=n$.

(2) $\sqrt{\{x^2-\frac{1}{2}\sqrt{(x^2+96)}\}}=x+1$.

(3) $(x-m)\sqrt{(an)}-(a-n)\sqrt{(mx)}=0$.

5. A detachment from an army was marching in regular column, with 7 men more in depth than in front, but the front being increased by 336 men, the detachment was drawn up in 5 lines. Find the number of men.

6. Shew that if $x+\sqrt{y}=a+\sqrt{b}$, $x=a$, $y=b$; and apply the principle to determine the square root of $12-4\sqrt{5}$.

Express $\frac{12-3\sqrt{3}}{12-4\sqrt{3}}$ as a fraction with a rational denominator.

7. What is meant by the root of an equation? One root of the equation $x^3-7x+6=0$ is 2, find the other two roots.

8. Solve the following equations :—

(1) $x^{-1}+x^{-\frac{1}{2}}=6$

(2) $xy+xy^2=12$, $x+xy^3=18$.

(3) $x+y=7$, $x^4+y^4=337$.

9. If x, y, z be in Geom. Prog., prove that

$$x^2y^2z^2(x^{-3}+y^{-3}+z^{-3})=x^3+y^3+z^3.$$

10. (1) Insert 5 arithmetical means between a and b. Sum the series

(2) $13, 12\frac{1}{4}, 11\frac{1}{2}$ to 9 terms.

(3) $\frac{10}{21}, \frac{5}{7}, 1\frac{1}{14}$, to 6 terms.

(4) $3\sqrt{2}, \sqrt{6}, \sqrt{2}$ to infinity.

11. When are magnitudes in harmonic progression? Find the harmonic mean between $m+n$ and $m-n$.

12. From a company of 50 men, 6 are draughted off every night on guard: on how many nights can a different guard be posted, and on how many of these will any one man be employed?

13. Solve the equations:—

(1) $x(2x+5)+3x+1=2x(3x+1)$.

(2) $x^2-xy=10$, $x^2+y^2=34$.

14. Find two numbers in the ratio of 5 to 9 such that the sum of their squares exceeds by 10 the square of their difference.

15. Sum the following series.

(1) $2+4+6+$ .... to 15 terms.

(2) $1+\frac{1}{2}+\frac{1}{4}+$ .... to infinity.

(3) $1-\frac{1}{5}+\frac{1}{25}-$ .... to infinity.

16. Solve the following equations:—

(1) $6x^2-7x=115$.

(2) $x^2+\frac{11}{2}x-4\sqrt{(2x^2+11x+5)}=2$.

(3) $x^2+3y^2=28$, $xy+y^2=12$.

17. Find two quantities in the ratio of 9 to 7, such that the sum of the squares is greater by 14 than the square of their difference.

18. Prove the formula for the sum of a geometrical progression, and apply it in finding the value of the recurring decimals ·77777 &c.

19. How many permutations can be formed of the letters in the word *permutations* ?

20. On how many nights may a different guard of 7 men be draughted from a company of 40 men? On how many of these would a particular man be taken?

21. Define a Logarithm and its Mantissa.

Shew that $\log_a mn = \log_a m + \log_a n$.

Find the logarithm of 243 to the base 9.

Given $\log 2 = \cdot 3010300$, find $\log 128$, $\log 125$ and $\log 256$.

22. Find the present value of ~~Rs~~ 1800 due 5 years hence at 4 per cent. compound interest. Having given,

$\log 18 = 1\cdot 2552725$, $\log 104 = 2\cdot 0170333$,

and $\log 1479\cdot 47 = 3\cdot 1701060$.

23. Solve the following equations.

(1) $\dfrac{1}{x-a} + \dfrac{1}{x-b} = \dfrac{a+b}{ab}$.

(2) $\dfrac{x}{y} - \dfrac{y}{x} = \dfrac{15}{4}$ and $x - y = 3$.

(3) $\dfrac{xy}{x^{\frac{1}{2}}y^{-\frac{1}{2}}} = 48$ and $\dfrac{xy}{\sqrt{x}} = 24$.

24. In a mixture of rum and brandy, the difference between the quantity of each is to the quantity of brandy as 100 is to the number of gallons of rum; and the same difference is to the quantity of rum as 4 is to the number of gallons of brandy. How many gallons are there of each?

25. If s, and s_1 be the sums to infinity and to n terms of a decreasing Geometric Progression whose first term is a;

show that $n \log \left(1 - \dfrac{a}{s}\right) = \log \left(1 - \dfrac{s_1}{s}\right)$

26. Find two numbers whose difference is 4, and the harmonic mean between them $7\frac{1}{2}$.

27. Form the quadratic equation whose roots are $a + \sqrt{b}$ and $a - \sqrt{b}$.

28. Expand $(3a-4x)^{\frac{4}{5}}$ to four terms; write down the $(r+1)^{th}$ term of $(1-x)^{-3}$.

29. Given $\log 2 = \cdot 3010300$, find $\log \sqrt[3]{\frac{625}{1024}}$ and $\log (\cdot 02)^{\frac{1}{100}}$.

30. Solve the following equations :—

(1) $\frac{2}{x-3}-\frac{3}{2x-1}=\frac{11}{7}$

(2) $\frac{a^2}{x-b}+\frac{b^2}{x-a}=a+b$

(3) $x^2-y^2=\frac{8}{3}xy, x+y=4.$

31. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the periphery of each wheel be increased one yard, it will make only 4 revolutions more than the hind-wheel in the same space. Required the circumference of each.

32. Sum the following series :—

(1) $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ to 7 terms.

(2) $\frac{1}{2}(2-\sqrt{3}), 1+\sqrt{3}, \frac{1}{2}(2+5\sqrt{3})$, etc. to 9 terms.

(3) $5, 4, 3\frac{1}{5}$, etc. to infinity.

33. If the first term of an arithmetic series be 1, the common difference 4, and the sum of the terms 120, determine the number of terms.

34. If $a=x^2-yz, b=y^2-xz, c=z^2-xy$, prove that

$$\frac{a^2-bc}{x}=\frac{b^2-ac}{y}=\frac{c^2-ab}{z}=(a+b+c)(x+y+z).$$

35. Given $3^{2x}+3^x=6$ and $4^{2x+y}-2\times 4^{x+y}=8$, find the values of x and y.

36. There are n arithmetical means between 1 and 31, and the 7^{th} mean : $(n-1)^{th}$ :: 5 : 9; find n.

37. Expand $(1-x)^{-7}$ to four terms; and find the coefficient of x^4 in $(1-x+x^2)^{-\frac{7}{2}}$.

38. Find the coefficient of x^6 in the expansion of $(1+2x+3x^2+\ldots\ldots)^{-8}$.

39. Given $\log 48=1{\cdot}6812412$ and $\log 18=1{\cdot}2552725$, find $\log 243$.

40. Find the value of a freehold estate yielding an annual income of Rs 1600, money making 5 per cent. per annum.

41. Solve the following equations :—

(1) $\dfrac{1}{2(x-3)}+\dfrac{3}{4(x-5)}=\dfrac{11}{12}$.

(2) $x^2+3xy=27,\ 2y^2+xy=14$.

(3) $\left.\begin{array}{l} x+4y+4\sqrt{x}=21+8\sqrt{y}+4\sqrt{(xy)}, \\ \text{and } \sqrt{x}+\sqrt{y}=6 \end{array}\right\}$

42. There is a field in the form of a rectangular parallelogram, whose length exceeds the breadth by 14 yards; and it contains 3200 square yards. Required the length and breadth.

43. Form the quadratic equation whose roots are $2+\sqrt{3}$ and $-1-\sqrt{3}$.

44. Sum the series

$$\left(\frac{n+1}{n-1}\right)^2,\ \frac{n+1}{n-1},\ 1,\ \frac{n-1}{n+1} \text{ to infinity.}$$

45. How many terms of the series 1, 3, 5, 7 &c. must be added together to produce the $(2r)^{\text{th}}$ power of a given quantity p.

46. $a \propto b+\dfrac{1}{c}$ and $b \propto c$; if $c=1$, $b=1$, then $a=1$; find the value of b when $a=4$, $c=2$.

47. Prove that the number of combinations of n things taken r together is the same as the number taken $n-r$ together. How many words can be made of all the letters of the word *rotation*.

48. Expand $(1+x+x^2)^{\frac{3}{2}}$ to 4 terms and in its expansion find the coefficient of x^{10}.

49. Given $\log 2={\cdot}3010300$, $\log 3={\cdot}4771213$. Find the value of x from the following equations.

(1) $10^{x^2}=2^{8x}$, (2) $2^{3x}\times 6^{3x-5}=3^{2x-2}\times 2^{5x}$.

50. Find the present value of an annuity of Rs 100 to begin immediately and to continue for 10 years reckoning compound interest at 5 per cent. Given $\log 1{\cdot}05 = {\cdot}0211893$ and $\log 644609 = 5{\cdot}8092963$.

51. Solve the following equations.

(1) $$\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{1}{x-a}-\frac{1}{x+b}\right)=\frac{a+b}{a-b}\times\frac{1}{x}.$$

(2) $$x^2+xy=3,\ y^2-xy=2.$$

(3) $$\sqrt{\left(x^2-\frac{1}{x^2}\right)}+\sqrt{\left(1-\frac{1}{x^2}\right)}=x^2.$$

(4) $$x^4-2x^2y+y^2=49.$$
$$x^4-2x^2y^2+y^4-x^2+y^2=20.$$

52. A person bought some cows for Rs 720; and found that if he had bought 6 more for the same money, he would have paid Rs 10 less for each. How many did he buy, and what was the price of each ?

53. Find a quadratic equation whose roots are the reciprocals of the roots of the equation $12x^2-25x+12=0$.

54. If the two roots of the equation $x^2+(a+b+c+d)x+(a+b)(c+d)=0$ be equal, prove that $(a+b)^2+(c+d)^2=2(a+b)(c+d)$.

55. Eliminate x and y from

$$\left.\begin{aligned} x+y+n&=0\\ (a+b)x+(b+c)y+(a+c)n&=0\\ x^2+y^2+n^2&=2(a^2+b^2+c^2-ab-ac-bc).\end{aligned}\right\}$$

56. If $S(n)=1+a+a^2+\ldots\ldots$to n terms.

$$S(-n)=1+\frac{1}{a}+\frac{1}{a^2}+\ldots\ldots\text{to } n \text{ terms.}$$

Prove that $S(n)=a^{n-1}\{S(-n)\}$.

57. The number of combinations of n things taken 4 together is to the number of combinations of $n-1$ things taken 5 together as 5 : 12; find n.

58. Find by the Binomial Theorem the $5th$ root of 30, correct to 4 decimal places.

59. Extract the $5th$ root of 64145, having given log 6·4145 = ·8071628, and log 9·15024 = ·9614326.

60. Find the amount of an annuity of Rs400 left unpaid for 10 years, reckoning compound interest at 4 per cent. per annum; having given log 104 = 2·0170333 and log 1480238 = 6·170333.

61. *Solve the following equations* :—

(1) $$\frac{a-b}{x-a}+\frac{b-a}{x-b}=\frac{a}{b}+\frac{b}{a}.$$

(2) $$(x+4)(x-3)+\sqrt{\{(x+3)(x-2)\}}=36.$$

(3) $$\left.\begin{aligned} &2x+y=26-7\sqrt{(2x+y+4)} \\ &\frac{2x+\sqrt{y}}{2x-\sqrt{y}}=\tfrac{13}{15}+\frac{2x-\sqrt{y}}{2x+\sqrt{y}} \end{aligned}\right\}$$

62. Form a quadratic equation whose roots are the squares of the roots of $x^2+2ax+a^2-b^2=0$.

63. If the roots of the equation $x^2-px+q=0$ be a and b, determine the equation whose roots are the arithmetic and geometric mean between a and b.

64. Solve

$$\left.\begin{aligned} &x+y+z=a+b+c \\ &\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3 \\ &\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\frac{2yz}{bc}=5 \end{aligned}\right\}$$

65. If the first term of an $A.P.$ be 1, the common difference 4 and the sum 120, find the number of terms.

66. The sum of the first and third terms of a geometric series is a, and of the second and fourth is b; find the first term.

67. Find by the Binomial Theorem the 5th root of $\frac{1000}{999}$ correct to 4 places of decimals.

68. Find the first 4 terms of the expansion of $(1+x)^{1+x}$.

69. From a company of 100 soldiers, a picket of 12 men is to be selected; determine in how many ways this can be done, (1) so as always to include two particular men, (2) so as always to exclude the same two men.

70. A person who receives an annuity of Rs500 puts it out always at simple interest at 5 per cent. on the day, that he receives it. In how many years will it amount to Rs6875.

71. Solve the following equations :—

(1) $x^2-ax+2a\sqrt{(x^2-ax+a^2)}=2a^2$.

(2) $x^3+4xy+y^3=40,\ x+y=4$.

(3) $\left.\begin{array}{l} xy+\sqrt{(xy)}=42 \\ 2x^2y+3\sqrt{(xy)}=324. \end{array}\right\}$

72. The equations $abx^2+acx+bc=0$ and $acx^2+bcx+ab=0$ have a common root, find the relation between a, b, c.

73. $A=ax+by+cz, B=ay+bz+cx, C=az+bx+cy$, prove that $A^3+B^3+C^3-3ABC=(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)$.

74. Eliminate x, y, z from the following equations :—

$$\left.\begin{array}{l} (a^2-1)(x+y)^2=a^2(x-y)^2 \\ (b^2-1)(y+z)^2=b^2(y-z)^2 \\ (c^2-1)(x+z)^2=c^2(x-z)^2 \end{array}\right\}$$

75. A person bought 20 goats and 12 sheep for 180 Rupees. He had 2 goats more for Rs27 than he had of sheep for Rs30. What was the price of each ?

76. Solve the following equations :—

(1) $x(x+1)+3\sqrt{(2x^2+6x+5)}=25-2x$.

(2) $(7+4\sqrt{3})x^2+(2+\sqrt{3})x=2$.

(3) $(1+x+x^2)(5-x-x^2)=9$.

77. If $x^2-yz=a^2, y^2-xz=b^2, z^2-xy=c^2$, prove that $(x+y+z)(a^2+b^2+c^2)=a^2x+b^2y+c^2z$.

78. Find the co-efficient of x^4 in the expansion of $(1+x+x^2+x^3)^{-3}$.

79. Eliminate x from

$$\left.\begin{array}{l} (a+b)x^2+(b+c)x+a+c=0 \\ (a+c)x^2+(a+b)x+b+c=0 \end{array}\right\}$$

80. The number of permutations of n things taken 5 together is 20 times the number taken 3 together ; find n.

81. Four numbers are in Geometrical Progression; the sum of the first and second numbers is 15, and the sum of the third and fourth is 30; find the numbers.

82. Find the value of $(1 \cdot 56)^{\frac{3}{2}} \times (1 \cdot 56)^{2} \times (1 \cdot 56)^{\frac{5}{2}} \times \ldots\ldots$ to 6 factors; having given $\log 156 = 2 \cdot 1931246$ and $\log 1536 \cdot 58 = 3 \cdot 1865559$.

83. Show that the co-efficient of x^{2n} in the expansion of $(1-x)^{-1} \times (1+x)^{-4}$ is $\frac{1}{6}(n+1)(n+2)(4n+3)$.

84. Find by the Binomial Theorem the value of $\left(\frac{1}{100002}\right)^{\frac{1}{5}}$ to 10 places of decimals.

85. Find the time in which a sum of money will double itself at $6\frac{1}{2}$ per cent. compound interest. Having given $\log 2 = \cdot 3010300$ and $\log 1 \cdot 065 = \cdot 0273496$.

86. Solve the following equations :—

(1) $x^2 - \frac{7}{4} = \frac{3}{4x}$.

(2) $\sqrt{\{ay(a-x)\}} = (a-b)x$ and $\sqrt{(ayx)} = (a-b)(a-x)$.

(3) $x^2+y^2+z^2=12n^2$

$xy+xz+yz=-6n^2$

$x+y=\frac{1}{2}(3-n)n$

87. Eliminate x from the equations

$$\left.\begin{array}{l}(a+b)x^2+(b+c)x+c+a=0\\(a-b)x^2+(b-c)x+c-a=0\end{array}\right\}$$

88. If $ax^2+by^2+cz^2+2cxy-2bxz-2ayz$ be resolvable into rational factors, find the relation between a, b and c.

89. Eliminate x, y, z from the equations :—

$$\left.\begin{array}{c}x^3+y^3+z^3=(x+y)(x+z)(y+z),\\a(x^2+y^2-z^2)=b(x^2+z^2-y^2)=c(y^2+z^2-x^2)\end{array}\right\}$$

90. If $x+y+z \propto x-y-z$ and $x^2+y^2+z^2 \propto x^2+y^2-z^2$, prove that $x \propto z$ and $y \propto z$.

91. Sum the series $a+(a-b)r+(a-2b)r^2+(a-3b)r^3+\ldots$ to n terms.

92. Solve the following equations :—

(1) $x^{\frac{7}{3}}=56x^{-\frac{2}{3}}+x^{\frac{5}{3}}$.

(2) $xyz=48,\ 12x=yz,\ 3xy=4z.$

(3) $\sqrt{\left(\frac{x+y^2}{4x}\right)}+\frac{y}{\sqrt{(y^2+x)}}=\frac{y^2}{x}\sqrt{\left(\frac{4x}{y^2+x}\right)}$,

$$\frac{\sqrt{x}+\sqrt{(x-y-1)}}{\sqrt{x}-\sqrt{(x-y-1)}}=y+1.$$

93. *A* and *B* gained by trading Rs 1000. Half of *A*'s stock was less than *B*'s by Rs 1000; and *A*'s gain was three-twentieths of *B*'s stock. What was the stock of each, and what were the respective shares of the gain?

94. A person makes a mixture of 76 gallons, consisting of brandy, rum and water, the quantities of which respectively are in geometrical progression. The number of gallons of brandy exceeds the number of gallons of water by 20. Required the quantities of each.

95. Find the present worth of Rs 1000 due 10 years hence at the rate of $4\frac{1}{2}$ per cent. compound interest.

Given log 1·045 = ·0191163 and log 643927 = 5·8088370.

96. Solve,
$$\left.\begin{array}{l} x^3+y^3+z^3=153, \\ x+z-y=3, \\ (x+z)\{3(x+z)+3+3xz\}=846-300y+30y^2. \end{array}\right\}$$

97. Find the co-efficient of x^4 in the expansion of $(1+x+x^2)^{-5}$.

98. If the coefficient of the $(r+1)$th term of the expansion of $(1+x)^{2n}$ be equal to that of the $(r+3)$th term, prove that $n=r+1$.

99. The sum of Rs 10000 is borrowed at 5 per cent. compound interest. After a certain time Rs 5816 was paid off by cash and for the balance a free-hold estate yielding an annual income of Rs 250 was given to the creditor. After what time from the date of borrowing was this transaction made?

Given log 1·05 = ·0211893, and log 1·0816 = ·0340667.

100. Find the difference between the present value of a free-hold estate, yielding an annual income of Rs 1000, to be occupied after 5 years and that of a similar estate to be occupied at once, money making 4 per cent. per annum compound interest. Given log 4 = ·6020600, log 1·04 = ·0170333. and log 205482 = 5·3127735.

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1861.

1. Extract the square root of $28-6\sqrt{3}$, and of $2\sqrt{(-1)}$.

2. Reduce the following expressions to their simplest forms :—

$$\frac{\cdot 6}{\sqrt{(1\cdot 5)}} \times \sqrt{\left(\frac{3-\frac{2}{9}}{4-2\frac{2}{3}}\right)}; \quad \frac{a^{\frac{3n}{2}}-a^{-\frac{3n}{2}}}{a^{\frac{n}{2}}-a^{-\frac{n}{2}}}; \quad \frac{x^{\frac{4}{3}}+a^{\frac{2}{3}}x^{\frac{2}{3}}+a^{\frac{4}{3}}}{x^{\frac{2}{3}}+a^{\frac{1}{3}}x^{\frac{1}{3}}+a^{\frac{2}{3}}}.$$

3. Solve the equations

(1) $(x-\frac{1}{2})(x-\frac{1}{3})+(x-\frac{1}{3})(x-\frac{1}{4})=(x-\frac{1}{4})(x-\frac{1}{6})$.

(2) $\left.\begin{array}{r}(x-y)(x^2+4y^2)=x^3+y^3\\ xy=c^2\end{array}\right\}$

4. Find the sum of n terms of a geometrical progression whose first term and common ratio are given.

If S_n denote the sum of n terms, find the sum of $S_1, S_2 \ldots S_n$.

5. Prove that there are as many combinations of n things taken r together, as there are, taken $n-r$ together.

If there be twenty pears at three a penny, how many different selections can be made in buying six penny worth? and in how many of these will a particular pear occur?

6. Write down the general term of the expression

$$\{xy-\sqrt{(3x^2y)}\}^{\frac{11}{3}}$$

1862.

1. Find the value of

$$\left(\frac{\sqrt{(27)}+\sqrt{(12)}}{\sqrt{(117)}+\sqrt{(52)}}\right)\left(\frac{\sqrt{(18)}+\sqrt{8}}{\sqrt{(63)}+\sqrt{(8)}}\right)$$ to three places of decimals.

2. Reduce the following expressions to their simplest forms :—

$$\frac{3x^2-8x+5}{x^3-4x^2+5x-2}; \quad \frac{2x^3+ax^2+4a^2x-7a^3}{x^3-7ax^2+8a^2x-2a^3}.$$

3. Eliminate a, b, c, from the equations

$$\frac{x^m}{a^m}+\frac{y^m}{b^m}+\frac{z^m}{c^m}=1=\frac{a^n+b^n+c^n}{p^n}$$

$$\frac{a^{m+n}}{x^m}=\frac{b^{m+n}}{y^m}=\frac{c^{m+n}}{z^m}.$$

4. Solve the equations

$$\frac{x-\sqrt{(2x)}+1}{x+\sqrt{(2x)}+1}=\frac{a}{b}$$

$$\left.\begin{aligned} x+y+z&=1\\ x^2+y^2+z^2+6xy&=0\\ \frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}&=0\end{aligned}\right\}$$

5. Write down the $(r+1)^{\text{th}}$ term of

$$\{(xy-\sqrt{(9yz)}\}^{\frac{17}{3}}.$$

1863.

1. Simplify :—

(a) $$\frac{x^2+(a-b)x-ab}{x^2+(a+b)x+ab}$$

$$\frac{e^{2x}x^2+e^{2x}-x^2-1}{e^{2x}x^2+2e^{x}x^2-e^{2x}-2e^{x}+x^2-1}$$

2. Solve the following equations :—

(a) $$\frac{198x+3}{4{\cdot}5x+3}+\frac{4x+5}{{\cdot}5x-1}-52=0$$

(b) $x^{\frac{1}{2}}+y^{\frac{1}{2}}=4$, and $x^{\frac{3}{2}}+y^{\frac{3}{2}}=28$.

(c) $(x+6)^2+2\sqrt{x(x+6)}=138+\sqrt{x}$.

3. Find the sum of the following series :—

(a) $\frac{1}{a}+\frac{2}{a}+\frac{3}{a}+$ &c. to 20 terms.

(b) $\frac{3}{2}-(\frac{3}{2})^{\frac{1}{2}}+1-$ &c. to n terms.

4. State the general rule called the Binomial Theorem, and illustrate it by giving the first four terms of the expansion of $(5-\frac{x}{6})^6$.

How many changes may be rung with four bells out of 7; how many with the whole peal?

1864.

1. Expand $(1+x)^{\frac{1}{5}}$ as far as x^4 by the Binomial Theorem.

2. The first term in an arithmetical progression is 1, the number of terms is 33; what must the common difference be, in order that the sum may be $149\frac{1}{2}$?

3. Suppose a body moves eternally in this manner, *viz.* 20 miles the first minute, 19 miles the second, $28\frac{1}{20}$th the third, and so on in geometrical progression; required the utmost distance it can reach.

4. Eliminate x from the quadratics,

$$Ax^2+Bx+C=0$$

$$A'x^2+B'x+C'=0$$

5. Two retailers jointly invest £700 in business; the one lets his money remain five months, the other only three, and each receives £450 capital and profit, how much did each advance at first?

6. A pays income tax at the rate of $7d.$ in the pound for three months, and for the remainder of the year at the rate of $5d.$; what should be his gross annual income to leave a surplus of £2000?

7. The fore-wheel of a carriage makes six more revolutions than the hind-wheel in 120 yds.; but the former would only make four more revolutions than the latter in the same distance if the spokes of each were lengthened by 5 inches; what is the circumference of each wheel.

1865.

1. Solve the equations :—

$$\left.\begin{aligned}(1+x)(1+y)&=10\\ x^2y+y^2x&=18\end{aligned}\right\}; \quad \frac{x^2}{y^2}+\frac{y}{x}+\frac{x}{y}=\frac{27}{4}-\frac{y^2}{x^2}, \quad x-y=2.$$

2. A bankrupt has three creditors, of whom A receives a shilling in the pound more than B or C, A is thus paid £100 less than B, and B £100 less than C; the total debts were £2600, and the assets £1200; how much was due to each.

3. Eliminate x from the following equations

$$x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)=m$$

$$x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)=n$$

4. Find the sum of the following series to n terms:—

$$a-ar+ar^2-ar^3+\&c.$$

Find the order of magnitude of the arithmetical, geometrical and harmonic mean between two numbers a and b.

5. Apply the binomial theorem to expand $\dfrac{3a}{(a^3-x^2)^{\frac{4}{3}}}$ in a series ascending by powers of x; obtain the first five terms of the series, and write down the $(r+1)$th term of the expansion.

1866.

1. Solve the equation:—

$$ax^2+2bx+c=0;$$

and determine the condition between the coefficients in order that——(i.) The roots may be possible

(ii.) The roots may be equal

(iii.) The roots may be equal in magnitude but of different signs.

Solve the equations

$$x=\cfrac{5}{6-\cfrac{5}{6-\cfrac{5}{6-x}}}$$

$$\left.\begin{aligned} x^2-y^2-z^2+2yz&=3\\ -x^2+y^2-z^2+2xz&=-9\\ -x^2-y^2+z^2+2xy&=-3 \end{aligned}\right\}$$

2. The men who were placed in the first division in the B. A. Examination of 1866, could have been arranged in two squares, the number in the front of one square being double the number in the front of the other square. The number of those who paid their fees, but were not present at the examination, was equal to the number in the smaller square; if another man had passed in the first division, the first division and the absentees would have together formed a square with one man more in the front than in the front of the larger square of the first division. If there had been one more man in the second division and one more rejected, the second division and the rejected men would have together formed a square with six men more in the front than in the front of the larger square of the first division, and the number of the second division would have been to the number rejected as 3 to 2. Find the number in each division.

3. The common ratio of a series in geometrical progression is 3; the sum of the first and third terms is equal to the sum of the squares of the first and second terms; find the sum of n terms. If $n=6$ show that the sum is 364.

4. *Find the number of permutations of* 7 *things taken* 4 together, and the number of combinations of 50 things 47 together.

Prove that the number of words which can be formed of the letters a, b, c, d, e, f, taken *three* together, each word containing one vowel *at least*, is 96.

5. Find, by the Binomial Theorem the 3*rd*, 6*th* and *rth* terms in the expansion of :— $\left(a^2x^2 - a^{-\frac{3}{5}}x^{\frac{9}{4}}\right)^{-\frac{15}{2}}$

1867.

1. Write down the roots of the equation $9x^2+15x-14=0$.

Solve the following :—

$$(1)\quad \frac{x+2a}{x-2a}=\left(\frac{x+a}{x-a}\right)^3$$

$$(2)\quad \left.\begin{array}{l} x^2+xy=a^2 \\ x^2-xy=b^2 \end{array}\right\}$$

2. Solve the equations :

$$\left.\begin{array}{l} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{array}\right\}$$

and eliminate the unknowns when $d_1=d_2=d_3=0$.

3. A number has three digits, the sum of which equals 10; the first and third exceed the second by 4, and the first and second exceed the third by 8. Find the number.

4. Shew how to sum a series in *A. P.* and insert three harmonic means between 4 and 2.

5. If $(x^2-y^2)z=(y^2-z^2)x$, show that x is to z in the duplicate ratio of x and y.

6. Find the number of Permutations of n things all together, whereof p are the same, q the same, r the same, and so on; $p+q+r+$.........being less than n.

How many diagonals can be drawn in a figure of n straight lines as sides.

7. Expand the series $(1+x+x^2)^{-3}$ to the fourth power of x inclusive, and extract the cube root of 1·03 to four places of decimals by the aid of the Binomial Theorem.

1868.

1. Solve the equations :—

$$77(x^2-1)=72x.$$

$$\left.\begin{array}{l}\dfrac{x}{a}-\dfrac{y}{b}=3\\[2ex]\dfrac{y}{b}-\dfrac{a}{x}=\frac{1}{3}\end{array}\right\}$$

Verify the answers to the pair of equations by inserting the roots in one of them.

2. A ploughing machine is engaged to plough a field at Rs 10 per bigah. After one day the machine is out of order, and the rest of the field is ploughed by a yoke of oxen in 4 days at the same rate per bigah. The machine can do in $1\frac{1}{4}$ days what 8 yoke of oxen can do in 5 days. The amount paid for the oxen was Rs 48 more than the number of bigahs. Find the number of bigahs.

3. If $S \propto a$ when b is constant, and $S \propto b$ when a is constant, show that $S \propto ab$ when neither a nor b is constant.

4. Find the sum of a geometric series of n terms. Sum ten terms of the series—

$$a+b+3a+2b+5a+4b+\&c.$$

If a be the first term and ar^{n-1} the last term of a geometric series, and ar^{n-1} be the first and a the last term of the same

series reversed, and if the terms of the first series, be divided by equidistant terms from the beginning of the second, then the sum of the resulting series will be $\frac{r^n - r^{-n}}{r - r^{-1}}$.

5. Find the number of permutations of n things taken r together.

How many arrangements may be made of all the letters of the word *Allahabad*?

6. State the Binomial Theorem; and prove it when the index is a positive integer.

Find the 9th term of $(\frac{1}{3}a - \frac{1}{2}b)^{12}$.

1869.

1. The expenses of a family when rice is at twenty seers for a rupee are fifty rupees a month; when rice is at twenty-five seers for a rupee the expenses are forty-eight rupees a month; what will they be when rice is at thirty seers for a rupee?

2. Solve the equations :—

(1) $$\frac{2x(x+1)}{x+2} - \frac{2x-7}{x-4} = 2(x-2).$$

(2) $$\left.\begin{array}{l} x^2 + xy = 12 \\ xy - y^2 = \ 2 \end{array}\right\}$$

If $\left.\begin{array}{l} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{array}\right\}$, show that

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}.$$

3. Prove that

$$(x^2 + y^2 + z^2)(a^2 + b^2 + c^2) - (ax + by + cz)^2$$
$$= (cy - bz)^2 + (az - cx)^2 + (bx - ay)^2$$

also show that if the first member of this identity $= 0$, then

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove that

$$\frac{x^2 + y^2}{a^2 + b^2} = \frac{y^2 + z^2}{b^2 + c^2} = \frac{z^2 + x^2}{c^2 + a^2} = \left(\frac{lx + my + nz}{la + mb + nc}\right)^2.$$

4. A travels at the rate of 3 miles an hour; B leaves the same place two hours after A and travels at the rate of 5 miles an hour; when and where will B overtake A?

5. Show how to find the sum of a series in A. P. Prove that the product of any three consecutive numbers is equal to the difference between the middle and its cube.

6. Assuming the number of permutations of n things r together; find the corresponding number of combinations.

Show how many diagonals can be drawn to a figure of fifteen sides.

1870.

1. Find the values of x, y, z from the equations

$$bz+cy=a$$
$$cx+az=b$$
$$ay+bx=c$$

Show that the result of eliminating a, b, c from the same equations is $x^2+y^2+z^2+2xyz=1$.

2. If α and β be the roots of $ax^2+bx+c=0$, prove that

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}.$$

Find the equation whose roots are the arithmetic and geometric means of the roots of the above quadratic.

3. The arithmetic mean of two numbers exceeds the geometric mean by $\frac{3}{2}$, and the geometric mean exceeds the harmonic mean by $\frac{6}{5}$; find the numbers.

4. Prove that the number of combinations of n things taken r at a time is the same as the number of them taken $n-r$ at a time.

How many different elevens can be formed out of thirteen cricketers? In how many of these will a particular man occur?

5. In a binomial expansion, prove that the co-efficient of the rth term from the beginning is equal to the co-efficient of the rth term from the end.

Write down the 19th term of $\left[2x^{\frac{1}{3}}-y^{\frac{1}{3}}\right]^{20}$.

6. Find how many years will elapse before a sum of money doubles itself at 8 per cent. compound interest; having given—

$$\log 2=0{\cdot}30103, \qquad \log 3=0{\cdot}47712.$$

1871.

1. Find the value of $a^4+3a^3b+4a^2b^2+3ab^3+b^4$ where a b are the *other* two roots of the equation,—

$$4x^3+(x-1)^3+2(x^2-1)=4$$

beside the root $x=1$.

2. Find accurately to four places of decimals the value of

$$\frac{1}{(1-\frac{1}{10})^{\frac{1}{10}}}.$$

3. If $x+y$ varies as z when y is constant, and if $x+z$ varies as y when z is constant, show that when *both* y and z vary, then $x+y+z$ varies as yz.

4. Find the value of x, y, and z from the equations

$$\frac{1}{a+b+c}=\frac{\frac{a}{x}+\frac{b}{y}}{c}=\frac{\frac{b}{y}+\frac{c}{z}}{a}=\frac{\frac{c}{z}+\frac{a}{x}}{b}.$$

5. A, B, C are three villages connected by straight roads; B lies due east and C due north of A, one mile further from A than B is. D is a village exactly half way on the road between C and B. A man walked from D to A in one hour, and afterwards proceeding at the same rate walked from A to B in 1 hour 12 minutes. Find the distance of each village from A.

1872.

1. Solve the equations :—

(1) $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0.$

(2) $\frac{1}{x-3}-\frac{2}{x-4}+\frac{1}{x-6}=0.$

(3) And show that if $\left.\begin{aligned} l_1x+m_1y+n_1z&=0\\ l_2x+m_2y+n_2z&=0\\ ax+by+cz&=d\end{aligned}\right\}$

then $\frac{x}{m_1n_2-m_2n_1}=\frac{y}{n_1l_2-n_2l_1}=\frac{z}{l_1m_2-l_2m_1}$

$$=\frac{d}{a(m_1n_2-m_2n_1)+b(n_1l_2-n_2l_1)+c(l_1m_2-l_2m_1)}.$$

2. Solve the quadratic $ax^2+2bx+c=0$, and show that its roots are equal if $b^2-ac=0$.

Hence show that the condition that $Ax^2+By^2+Cz^2+2Dyz+2Ezx+2Fxy$ splits up into factors is that:—

$$AD^2+BE^2+CF^2-2DEF-ABC=0.$$

3. Find the sum of n terms of an $A.\ P.$, whose common difference is b, and first term a.

A class consists of a number of boys whose ages are in $A.\ P.$, the common difference being four months. If the youngest boy is just eight years old, and the sum of the ages is 168 years, find the number of boys in the class.

4. Find the number of combinations of n things r together where they are all different, and show that it equals the number of combinations of n things $n-r$ together.

5. Prove the Binomial Theorem for a positive integral exponent. Evaluate $\sqrt{(24)}$ by means of the Theorem to five places of decimals.

1873.

1. Simplify

$$\frac{\dfrac{y^4-x^4}{x^4+x^2y^2+y^4}+\dfrac{x^2}{x^2-xy+y^2}-\dfrac{y^2}{x^2+xy+y^2}}{\dfrac{x^3+y^3}{x^3-y^3}-\dfrac{x^3-y^3}{x^3+y^3}}\times\frac{x-\dfrac{y^2}{x-\dfrac{y^2}{x+\dfrac{y^2}{x}}}}{y+\dfrac{x^2}{y-\dfrac{x^2}{y+\dfrac{x^2}{y}}}}$$

and find the value of—

$$\frac{a(m+n)-bm-cn}{x}+\frac{b(n+l)-cn-al}{y}+\frac{c(l+m)-al-bm}{z}.$$

given that—

$$\frac{\dfrac{x}{a}+\dfrac{y}{b}-\dfrac{2}{3}}{\dfrac{1}{a}+\dfrac{1}{b}}=\frac{\dfrac{y}{b}+\dfrac{z}{c}-\dfrac{2}{3}}{\dfrac{1}{b}+\dfrac{1}{c}}=\frac{\dfrac{z}{c}+\dfrac{x}{a}-\dfrac{2}{3}}{\dfrac{1}{c}+\dfrac{1}{a}}$$

and $lx+my+nz=0$.

2. Solve the equations :—

(1) $$\left.\begin{array}{l} x^2+2y^2+3z^2=23 \\ 3x+y-5z=0 \\ 7x-3y-9z=0 \end{array}\right\}$$

(2) $\sqrt{(5-x)}+\sqrt{(x+8)}=5\sqrt{(2-x)}$.

(3) $2(x^2-3x+1)^2+5(x^2-3x+1)+3=0$.

Write down the sum and product of all the roots of the equation, $x^3-7x^2-8x=0$.

3. Find the sum of n terms of a geometrical progression.

The first term of an $A. P.$ is the same as that of a $G. P.$, the common difference of the one and the common ratio of the other are both 2; and the sum of 5 terms of each series is the same. Find the 5th term of each series.

4. Prove that the number of permutations of n things taken r together is $n(n-1)\ldots\ldots\ldots\ldots\ldots\ldots(n-r+1)$.

Ten candidates pass an examination in the first division. Three scholarships of different values are to be given among these men irrespective of their place in the examination. How many different scholarship's lists could be made? How many could be made if the scholarships were of equal value?

5. Assuming that—

$$(x+a_1)(x+a_2)\ldots\ldots(x+a_n)=x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots$$
$+p_{n-1}x+p_n$, when p_r = sum of the products of $a_1, a_2, a_3\ldots\ldots a_n$ taken r together, prove the binomial theorem for a positive integral index.

Find the first four terms and the greatest numerical coefficient in the expansion of $\left(3ax-\dfrac{2b^3}{x}\right)^{14}$.

Write down the coefficient of x^{10} in the expansion of the tenth root of $1+x$.

Find as far as three places of decimals the value of $(1-\frac{3}{10})^{-\frac{2}{3}}$.

1874.

1. A bag contains 160 coins consisting of half-crowns, shillings, sixpences, and fourpences, and the values of the sums of money represented by each denomination of coin are the same; how many of each are there?

2. What is the difference between the factors and terms of an algebraical expression? What is the meaning of the expression $a^2-(b-c)d$? Show that its value is $a^2-bd+cd$.

Show that when $A_0x^n+A_1x^{n-1}+A_2x^{n-2}+\&c.+A_{n-1}x+A_n$ is divided by $x-a$, the remainder is $A_0a^n+A_1a^{n-1}+\&c.+A_{n-1}a+A_n$. Hence determine when $x^n \pm a^n$ is divisible by $x \pm a$.

3. Show that if $2s=a+b+c$, then—

$$abc-a(s-b)(s-c)-b(s-a)(s-c)-c(s-a)(s-b)=2(s-a)(s-b)(s-c).$$

4. If α, β be the roots of the equation $x^2+px+q=0$, then $(x-\alpha)(x-\beta)=x^2+px+q$ for all values of x.

Solve the equations—

$$(\alpha)\quad x^4+2ax^3=2x+\frac{1}{a^2}.$$

$$(\beta)\quad xy=c(x+y)\,;\; xz=b(x+z)\,;\; yz=a(y+z).$$

5. If A, G, H be the arithmetic, geometric and harmonic means between any two quantities, then $G^2=AH$.

The square of the arithmetic mean between two numbers exceeds that of the geometric by 400, and the square of the geometric exceeds that of the harmonic by 144; what are the numbers?

6. Find the coefficient of x^n in the expansion of $\dfrac{a+bx^2}{(1-cx)^8}$.

1875.

1. Solve the equations—

$$\text{(i.)}\quad \left.\begin{aligned} &\frac{x-a}{b-c}+\frac{y-b}{c-a}+\frac{z-c}{a-b}=0. \\ &ax+by+cz=0. \\ &\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0. \end{aligned}\right\}$$

$$\text{(ii.)}\quad \frac{3}{5}\left(\frac{x-1}{2}\right)+2\left(\frac{x-2}{6}\right)+\frac{5}{8}\left(\frac{x-3}{4}\right)=3\tfrac{1}{30}.$$

$$\text{(iii.)}\quad \frac{x^3+y^3}{x-y}-\frac{x^3-y^3}{x+y}=\frac{x^2+y^2}{\frac{1}{y}-\frac{1}{x}}\,;\quad \frac{\frac{x}{y}+\frac{y}{x}+1}{\frac{x}{y}+\frac{y}{x}-1}=\frac{2(x^3-y^3)}{x^2+y^2}.$$

2. Write down the roots of the equation $ax^2+bx+c=0$: and hence deduce that (i.) the sum of the roots $=-\frac{b}{a}$, and (ii.) the product of the roots $=\frac{c}{a}$.

What relations will hold between the roots if (i.) $b=0$ or if (ii.) $c=-a$.

Find all the roots of the equation—

$x^2(x^2-3x+2)-7x(x^2-3x)+12x^2-50x+24=0.$

3. Show how to find the sum of a geometrical progression to n terms and also *ad infinitum* when the series is convergent.

Three quantities x, y, z, are in arithmetical progression. They are also in harmonic progression. Prove that they are in geometrical progression.

4. Write down all the variations of the four letters a, b, c, d, taken three together in which c stands first.

Find how many words could be made from the letters of the word *Orion*, supposing (i) the letters may stand in any order, and (ii) supposing that two consonants may not stand together.

If $f(m)=1+mx+\frac{m(m-1)}{1\cdot 2}x^2+$ &c., show by actual multiplication as far as the third power of x in the product, that $f(m)\times f(n)=f(m+n)$.

Assuming that $f(m)\times f(n)\times f(p)\ldots\ldots=f(m+n+p\ldots)$ prove the binomial theorem for a positive fractional index.

Evaluate the seventh root of 127 to five places of decimals by the binomial theorem.

1876.

1. Prove in any way that—

$$\frac{3\sqrt{8}+2\sqrt{7}}{\sqrt{8}+\sqrt{7}}=2\cdot 51\ \ldots\ .$$

2. What is meant by the Highest Common Divisor or G. C. M. of two algebraic expressions?

Find the H. C. D. of $20x^4+x^2-1$ and $25x^4+5x^3-x-1$.

3. If α and β are the roots of the equation $ax^2+bx+c=0$, show that the expression $ax^2+bx+c=a(x-\alpha)(x-\beta)$. Form a quadratic equation one of whose roots shall be $\sqrt{5}+1$.

4. Solve the equations—

(1) $$\frac{(x-a)(x-b)}{(x-a-b)}=\frac{(x-c)(x-d)}{(x-c-d)}.$$

(2) $x+y+x^2+y^2=18$, $xy=6$.

(3) $$\frac{xy}{x+y}=1, \frac{yz}{y+z}=2, \frac{xz}{x+z}=3.$$

5. The latter half of $2n$ terms of any arithmetical series $=\frac{1}{3}$rd of the sum of $3n$ terms of the same series. Prove it.

Sum the series—

$$3-1+\tfrac{1}{3}-\tfrac{1}{9}+\&c. \text{ ad inf.}$$

$$1+3+7+15+\&c. \text{ to } n \text{ terms.}$$

6. Prove that the number of permutations of n things taken all together, of which p are alike, q are alike, and r are alike, is—

$$\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}.$$

How many different words can be formed out of the letters of the word Constantinople? In how many of these will the three n's be consecutive letters?

7. What is the coefficient of x^m in the expansion of $(x+y)^n$?

Prove that—

$$1+n+\frac{n(n-1)}{1\cdot2}+\&c.+n+1=2^n.$$

Write down the coefficients of x^{17} and x^{18} in the expansion of $(a^4-bx^3)^{10}$.

Give the first four terms of the expansion of $\dfrac{1}{(2-3x^2)^{\frac{4}{5}}}$.

1877.

1. Simplify $\dfrac{3\sqrt{2}}{\sqrt{6}-\sqrt{3}}-\dfrac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}+\dfrac{2\sqrt{3}}{\sqrt{6}+2}$, and prove that

$$\left\{\frac{1+\sqrt{(-3)}}{2}\right\}^6=1.$$

2. Solve the equations :—

(1) $\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{6}$.

(2) $\left.\begin{array}{l}3x^2+2y^2=50\\ xy-3y^2=1\end{array}\right\}$

(3) $(x+4)(x+1)+\sqrt{\{(x+5)(x-3)\}}=3x+31$.

3. If α and β be the roots of the equation $x^2-px+q^2=0$, prove that $\alpha+\beta=p$, and $\alpha\beta=q^2$.

If x be *real*, prove that p cannot lie between $+2q$ and $-2q$.

Show that the roots of $x^2-(p^2-3q^2)x+q^2(p^2-4q^2)=0$ are the product and the square of the difference of the roots of the original equation.

4. Define geometrical progression.

Prove that the $2n^{\text{th}}$ term of any geometrical series is the mean proportional between the n^{th} and $3n^{\text{th}}$ terms.

s_1, s_2, s_3, are the sums to n terms, $2n$ terms, and to infinity of a G. P.; show that

$$s_1(s_1-s_3)=s_3(s_1-s_2).$$

Sum the series

$$2+5+10+17+\ldots\ldots\text{to } n \text{ terms.}$$

5. When is x said to vary directly as y and inversely as z? If x varies directly as y and inversely as z, and $x=a$ when $y=b$ and $z=c$; find the value of x when $y=b^2$ and $z=c^2$.

6. A father with eight children takes them three at a time to the Zoological Gardens, as often as he can without taking the same three children together more than once. How often will he go, and how often will each child go?

7. Write down the first four and the r^{th} terms of the expansion of $(1-x)^n$, and show that the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms.

Prove that

$$(1+x+x^2+\ldots\text{to}\propto)\ (1+2x+3x^2+\ldots+\text{to}\propto)$$
$$=\tfrac{1}{2}(1\cdot2+2\cdot3x+3\cdot4x^2+\ldots+\text{to}\propto)$$

8. Find the amount of a given sum in any time at compound interest.

Find in how many years Rs 100 will become Rs 1000 at 4 per cent. compound interest; having given log 104 = 2·0170333. Give the answer correct to 2 places of decimals.

1878.

1. Reduce to a decimal the fraction $\frac{3+\sqrt{(18)}-\sqrt{(50)}}{\sqrt{(98)}-\sqrt{(72)}-1}$.

2. What is meant by the G. C. M. of two algebraical expressions?

Find the L. C. M. of $6x^4-7x^2+2$, and $2x^3+6x^2-x-3$.

3. Solve the equations :—

(1) $\frac{x+6}{x+7}-\frac{x+1}{x+2}=\frac{1}{3x+1}$;

(2) $3x^2-4xy+5y^2=33$, $4x^2-xy=10$;

(3) $x+y+z=3, yz+xz+xy=-1, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{3}$.

4. If α and β be the roots of the equation $x^2+2ax+b=0$, form a quadratic equation with rational coefficients, one of whose roots is $\alpha+\beta+\sqrt{(\alpha^2+\beta^2)}$.

5. If $2s=a+b+c$, prove that

$$(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$$
$$=\tfrac{1}{2}(a^3+b^3+c^3-3abc).$$

6. The sum of n terms of an A. P. is $2n^2$; find the first term and the common difference.

Sum to n terms, and, if possible, to infinity, the series

(1) $6-3+\frac{3}{2}-\frac{3}{4}+\ldots\ldots\ldots\ldots\ldots$

(2) $2+7+14+23+34+\ldots\ldots\ldots$

7. For what value of r is the number of combinations of n things taken r at a time greatest?

If C_r denote the number of combinations of n things taken r at a time, show that $C_0+C_2+C_4+\ldots\ldots=C_1+C_3+C_5+\ldots\ldots$

8. Write down the coefficients of x^6 and x^{10} in the expansion of $(a^2+3bx^2)^{-5}$, and the rth term of the expansion of $(1-x)^{\frac{3}{2}}$.

1879.

1. At what rate per cent., compound interest, will £175 amount to 192*l.* 18*s.* 9*d.* in two years?

2. Find the value of $\left(\dfrac{2+\sqrt{12}-\sqrt{27}}{2+\sqrt{48}-\sqrt{27}}\right)^{\frac{1}{3}}$ to three places of decimals.

3. Solve the equations—

(1) $\dfrac{x-3}{2}+\dfrac{3x-2}{8}=\dfrac{2x}{5}+\dfrac{x+5}{10}$;

(2) $\dfrac{1}{x^2}+\dfrac{1}{y^2}=13,\ \dfrac{1}{x}-\dfrac{1}{y}=1$;

(3) $3z-2y=15,\ z-2x=1,\ x+y+z=25.$

4. If one root of the equation $ax^2+bx+c=0$ be the square of the other, prove that $b^3+a^2c+ac^2=3abc.$

5. Sum the series—

(1) $1^2+3^2+5^2+7^2+$............to n terms.

(2) $48-36+27-20\frac{1}{4}+$...to n terms and to infinity.

6. How many words, each consisting of two vowels and two consonants, can be made out of the letters of the word 'devastation'? In how many of them will the two t's be together?

7. Write down the rth terms in the expansions of $(4x-3y)^n$ and $(1-3x^2)^{\frac{3}{7}}$, and the coefficients of x^{11} and x^{12} in the expansion of $(1+x+x^2+x^3+\text{...to infinity})^{-11}$.

1880.

1. Find the value correct to three places of decimals of—

$$\frac{\sqrt{(\cdot05-\cdot005)}}{\sqrt{(\cdot05)}-\sqrt{(\cdot005)}}.$$

2. Prove that—

$$\frac{a^2\left(\frac{1}{b}-\frac{1}{c}\right)+b^2\left(\frac{1}{c}-\frac{1}{a}\right)+c^2\left(\frac{1}{a}-\frac{1}{b}\right)}{a\left(\frac{1}{b}-\frac{1}{c}\right)+b\left(\frac{1}{c}-\frac{1}{a}\right)+c\left(\frac{1}{a}-\frac{1}{b}\right)}=a+b+c.$$

3. Solve the following equations :—

(1) $\dfrac{(x+a)(x+mb)}{(x-ma)(x-b)}=\dfrac{(mx+a)(x+b)}{(x-a)(mx-b)}.$

(2) $$\left.\begin{array}{l} x+y+z=a+b+c \\ x^2+y^2+z^2=a^2+b^2+c^2 \\ \dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=3. \end{array}\right\}$$

4. Prove that the sum of the roots of the equation $x^2+px+q=0$ is equal to $-p$ and their product to q.

If $a\pm\sqrt{\beta}$ be the roots, prove that $\dfrac{1}{a}\pm\dfrac{1}{\sqrt{\beta}}$ will be the roots of the equation $(p^2-4q)(p^2x^2+4px)=16q$.

5. Find the number of permutations of n things, taken r together. In how many of these will three given things occur?

6. Find the sums of the following series:—

(1) $(x+y)^2+(x^2+y^2)+(x-y)^2+$&c. to n terms.

(2) $1+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+$&c. to n terms.

(3) $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+$&c. to n terms.

7. State the Binomial Theorem.

Employ it to show that $\sqrt[3]{(1010)}=10\cdot033+$

Find the first two terms in negative powers of x of the expansion of

$$\frac{x-h}{\{(x-h)^2+k^2\}^{\frac{3}{2}}}-\frac{x+h}{\{(x+h)^2+k^2\}^{\frac{3}{2}}}$$

where x is large compared with either h or k.

Prove that—

$$\left(\frac{1+x}{1-x}\right)^n=1+n\frac{2x}{1+x}+\frac{n(n+1)}{1\cdot2}\times\frac{2^2x^2}{(1+x)^2}+\&c.$$

1881.

1. If one year be added to the tenth part of the sum of the squares of the ages of two brothers the total will be seven times the difference between their ages; and next year the elder will be half as old again as the younger. What are their ages?

2. Find the sum of n terms of an Arithmetical progression of which the first term, the common difference and the number of the terms are given.

Find the sum of 20 terms of the series—

$$3+4+8+9+13+14+18+19+\&c.$$

3. Show how to find the greatest term in the expansion of $(x+a)^n$, when n is a positive integer.

What is the coefficient of x in the expansion of

$$\frac{1+4x^2+x^4}{(1-x)^4}?$$

4. Find the present value of an annuity of A pounds to commence at the end of P years and then to continue Q years.

Find the present value of a perpetual annuity of 729 £, the first payment of which is to be made in three years' time, interest being reckoned at 8 per cent.

1882.

1. Prove $a\times b=b\times a$. And solve the equations :—

(i) $$\frac{1}{\sqrt{x}-\sqrt{(x-2)}}+\frac{1}{\sqrt{x}+\sqrt{(x+2)}}=1.$$

(ii) $4x^4-16x^3+23x^2-16x+4=0.$

2. Prove the rule for finding the G.C.M. of two algebraical expressions.

Find the G.C.M. of $15x^3-4x^2-53x+30$
and $15x^3-x^2-31x-15$.

3. What is a "surd"? Prove that if two quadratic surds cannot be reduced to others which have the same irrational part, their product is irrational.

$$\text{Simplify } \sqrt{(18)}+\frac{8}{\sqrt{2}}+\frac{\sqrt{(24)}}{3\sqrt{3}}.$$

4. If $a:b$ be a ratio of less inequality, and x a positive quantity, the ratio $a+x:b+x$ is greater than the ratio $a:b$.

If $x:y$ be the ratio $a:b$ in its lowest terms, prove that

$$\frac{x+1}{y+1}>\frac{a+1}{b+1} \text{ if } b>a.$$

5. A shopkeeper has one maund of rice: as soon as he has sold four seers he mixes with the remainder four seers of an

inferior kind. How often must he repeat this process that half of the whole may be of the inferior kind? Given $\log_{10}2 = \cdot 30103$: $\log_{10}3 = \cdot 47712$.

6. Prove the Binomial Theorem for a positive integral index.

If $(1+x)^n = p_0 + p_1x + p_2x^2 + \dots + p_nx^n$, find the value of

$$p_1 + 2p_2 + 3p_3 + \dots + np_n.$$

1883.

1. Calculate the value of $\sqrt{(9-4\sqrt{5})}$ correctly to two places of decimals.

Simplify $(\sqrt{6}-1)(\sqrt{3}+2)(2\sqrt{2}-\sqrt{3})$.

If $\left(x+\frac{1}{x}\right)^2 = 3$, prove that $x^3 + \frac{1}{x^3} = 0$.

2. Solve the equations:—

(i) $\dfrac{x^2-a^2-b^2}{c^2} + \dfrac{c^2}{x^2-a^2-b^2} = 2.$

(ii) $\left.\begin{array}{l}\sqrt{(x-y)} = \sqrt{(y+1)}\\ x+y=6\end{array}\right\}$

3. Prove that the sum of the roots of the equation

$$ax^2+bx+c=0 \text{ is } -\frac{b}{a}, \text{ and their product } \frac{c}{a}.$$

If r be the ratio of the roots shew that $\dfrac{(r+1)^2}{r} = \dfrac{b^2}{ac}$.

4. Find the sum of n terms of the series a, ar, ar^2, &c.

If S be the sum, P the product, and R the sum of the reciprocals of the n terms of the same series, prove that $P^2 = \left(\dfrac{S}{R}\right)^n$.

5. Shew that the number of combinations of n things taken r together is the same as the number of combinations of n things taken $(n-r)$ together.

6. A committee of 7 members is to be chosen out of 20 municipal commissioners, of whom 15 are Hindoos and 5 Mahomedans, in such a way that 5 Hindoos and 2 Mahomedans shall

be on the Committee. In how many different ways can such a committee be constituted, and from how many of these will a particular Hindoo Commissioner be excluded.

7. Write down the expansion of $(x+y)^n$, and the coefficient of x^{11} in $(x-2y)^{13}$.

8. Find the amount of an annuity left unpaid for any number of years, allowing compound interest.

1884.

1. Find the value of $\frac{1}{2} \cdot \frac{(x-1)^2}{x+1} + \frac{7}{3} \cdot \frac{x+1}{x+2}$

when $x = \sqrt{(3)} - 1$.

Extract the square root of $11 - 6\sqrt{(2)}$ correct to two places of decimals.

2. Solve

$$(1) \quad x + \frac{1}{y} = a \qquad (2) \quad x^2 + y^2 + z^2 = 84$$

$$y + \frac{1}{x} = b \qquad x + y + z = 14$$

$$xz = y^2$$

3. Distinguish between an equation and an identity, prove that a quadratic equation cannot have more than two different roots, and *hence* show that

$$\frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-c)(x-a)}{(b-c)(b-a)} + \frac{c^2(x-a)(x-b)}{(c-a)(c-b)}$$

in its simplest form is x^2.

4. Find the n^{th} term of the following series and the sum to n terms

$$(1) \quad 9 + \frac{25}{3} + 7\frac{2}{3} + \&c.$$

$$(2) \quad 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \&c.$$

5. Show that the coefficient of x^r in the expansion of

$(1+x)^{\frac{10}{3}}$ is $\frac{10}{3r} \cdot \frac{1.4.7....(3r-8)}{\underline{|r}} \cdot (-1)^{r-2}$.

1885.

1. Divide—

$$2x^3-6x+5 \text{ by } \sqrt[3]{(2)}\,x+\sqrt[3]{(4)}+1.$$

If $x-a$ divide $x^n+p_1\,x^{n-1}+\ldots\ldots+p_n$ without remainder, prove that $a^n+p_1\,a^{n-1}+\ldots\ldots\ldots+pn=0$.

2. Solve the equations :—

$$\text{(i.)}\quad \frac{x+2}{1+2x}+\frac{3x+4}{3+4x}=\frac{x-2}{1-2x}+\frac{3x-4}{3-4x}.$$

$$\text{(ii.)}\quad y+z : z+x : x+y :: a : b : c$$

$$(y+z)^2+(z+x)^2+(x+y)^2=1.$$

3. If α and β be the roots of the equation $x^2+px+q=0$, prove that $\alpha+\beta=-p$ and $\alpha\beta=q$.

If the ratio of the roots of the equation $x^2+px+q=0$ be equal to that of the roots of $x^2+p_1x+q_1=0$, shew that $p^2 . q_1=p_1{}^2 . q$.

4. Prove that the number of combinations of $x+y$ things taken x together is equal to the number taken y together.

If the number of combinations of n things taken r together be equal to the number taken s together, either—

$$r=s \text{ or } r+s=n.$$

5. Investigate the order of magnitude of the Arithmetical, Geometrical, and Harmonical mean between two given numbers.

6. Write down the r^{th} term of the expansion of $(a-x)^{-\frac{1}{4}}$.

1886.

1. Express—

$$\frac{m(m-1)(m-2)(m-3)}{n(n-1)(n-2)(n-3)}$$

as a fraction with a rational denominator when

$$m=\sqrt{(-3)}+1,\ n=\sqrt{(-2)}+1.$$

Find the value of—

$$x^4+4x^3+6x^2+4x+9, \text{ when } x=\sqrt{(-2)}-1.$$

2. Solve the equations :—

(1) $3x+2\sqrt{(x^2-3x+9)}=x^2+6.$

(2) $\begin{cases} x^2+xy=(a-b)^2. \\ xy+y^2=4ab. \end{cases}$

(3) $\begin{cases} x^5-y^5=31. \\ xy=2. \end{cases}$

Find five values of x only.

3. The roots of x^2-px+q are α, β; form the equation whose roots are $\frac{2}{\alpha}$, $\frac{2}{\beta}$ and prove

(1) $p=\alpha+\beta.$ (2) $\frac{1}{\alpha^3}+\frac{1}{\beta^3}=\frac{p^3}{q^3}-\frac{3p}{q^2}.$

4. Find the n^{th} term in each of the following series, and the sum of (2) to infinity.

(1) $\frac{1}{n}+\frac{n+1}{n}+\frac{2n+1}{n}+\&c.$

(2) $\sqrt{(3)}+\frac{1}{\sqrt{(3)}}+\frac{1}{3\sqrt{(3)}}+\&c.$

(3) $4+4\frac{2}{7}+4\frac{8}{13}+5+\&c.$

5. Find by multiplication or otherwise the cube of

$$1+\frac{1}{3}x+\frac{1.4}{3.6}x^2+\frac{1.4.7}{3.6.9}x^3+\&c.,$$

as far as x^3 in its simplest form, and then show by the Binomial theorem that the $(n+1)^{\text{th}}$ term is

$$\frac{1.4.7\ldots\ldots(3n-2)}{3.6.9\ldots\ldots\ldots3n}\cdot x^n.$$

6. Find $\frac{1}{e^{\frac{1}{2}}}$ correct to two places of decimals.

B. C. E. EXAMINATION PAPERS.

1862.

1. Solve the following equations :—

$$\left.\begin{aligned}\frac{x}{y}-\frac{y}{x}&=\frac{x+y}{x^2+y^2}\\ \frac{x^2}{y^2}-\frac{y^2}{x^2}&=\frac{x-y}{y^2}\end{aligned}\right\}\dots\dots\dots\dots(a)$$

$$\left.\begin{aligned}x+y+z&=\frac{19}{6}\\ x^2+y^2+z^2&=\frac{133}{36}\\ y^2&=xz\end{aligned}\right\}\dots\dots\dots\dots(b)$$

$$\left.\frac{y^2}{2}=\frac{5x+4}{3}=\frac{4x-4}{2}\right\}\dots\dots\dots\dots(c)$$

2. If A varies as B, AC varies as D^2 and B^2 varies as CD, then $(A+B+C)^4$ varies as $ABCD$ or varies as $A^4+B^4+C^4$.

3. Divide $x^{\frac{3}{2}}-\frac{1}{2}x^{\frac{1}{2}}$ by $x^2+\frac{1}{2}$, to 4 terms, and cube the expression $\{x+\sqrt{(-y^2)}\}^{\frac{1}{3}}-\{x-\sqrt{(-y^2)}\}^{\frac{1}{3}}$.

4. Sum the following terms :—

(*a*) $1^3+2^3+3^3$ to n terms.

(*b*) $1\times 2x+2\times 4x^2+3\times 8x^3+\&c.$ to infinity.

5. What will £50 amount to at 4 per cent. compound interest, in 10 years, interest paid half-yearly.

1863.

1. Prove that

$$x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz=(y+z)(z+x)(x+y).$$

2. Define the terms "greatest common measure" and "least common multiple."

Reduce to its lowest terms the fraction

$$\frac{x^5+11x^3-54}{x^5+11x+12}.$$

3. Solve the equations

(1) $\dfrac{(x+a)(x+b)}{x+a+b} = \dfrac{(x+c)(x+d)}{x+c+d}$.

(2) $(4x+2)^4 - (3x+1)^4 = (2x+4)^4 - (x+3)^4$.

4. Find the sum of n terms of a geometrical progression, of which the first term and common ratio are given. Prove that no two unequal numbers can have their arithmetical and geometrical means equal.

5. Find the rth term of the expansion of $(a - a^{-1}x^{-1})^{-2}$; and hence deduce the first five terms of the series.

6. A man starts from the foot of a mountain to walk to its summit; his rate of walking during the second half of the distance is half a mile per hour less than his rate during the first half, and he reaches the summit in $5\frac{1}{2}$ hours. He descends in $3\frac{3}{4}$ hours by walking at a uniform rate which is one mile per hour more than his rate during the first half of the ascent. Find the distance of the summit and his rate of walking.

1864.

1. Find the sum, difference, and product of sum and difference of the following quantities $\dfrac{a+b}{a-b}$; $\dfrac{2ab}{a^2-b^2}$.

2. Solve the following equations.

(a) $\left.\begin{aligned} \frac{x}{3}+\frac{y}{5}&=8. \\ \frac{x}{9}-\frac{y}{10}&=1. \end{aligned}\right\}$

(b) $3(x-2)^2 = 18 + (8x+1)$.

3. Sum the following series.

(a) $1 - \frac{2}{3} + \frac{4}{9} -$ &c. to n terms.

(b) $1 - 2x + 3x^2 -$ &c. to infinity.

4. Insert five mean proportionals between 8 and 27 to four places of decimals.

1865.

1. Find the G. C. M. of x^3+4x^2+2x-1 and x^3+2x^2-4x+1.

If $a : b :: c : d$, prove that
$a+b : a-b :: \sqrt{(ac)}+\sqrt{(bd)} : \sqrt{(ac)}-\sqrt{(bd)}$.

2. Given the first term and common ratio of a geometrical progression, find the nth term and the sum of n terms.

Sum the series

$1-\frac{2}{3}+\frac{4}{9}-$&c.... ... ... to n terms.
$\frac{4}{15}+\frac{1}{5}+\frac{3}{20}+$&c. ... ... to infinity.

3. Find the coefficient of x^r in the expansion of $(1-2x)^{-1}$.

4. Solve the following equations :—

$$x^2+2x=12-4\sqrt{(x^2+2x)}$$

$$\left.\begin{array}{l} x^2-xy+y^2=7 \\ x^2+xy+y^2=19 \end{array}\right\}$$

5. Two railway trains moving in opposite directions pass each other somewhere between two stations A and B. One train leaves B at 10 minutes past 4 and arrives at A 20 minutes to 5. The other leaves A at 10 minutes past 4 and arrives at B at 20 minutes to 5. Find the time at which they meet.

1866.

1. Find the L. C. M. of $x^2-10x+24$, $x^2-8x+12$ and x^2-6x+8.

2. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each is equal to $\frac{ma+nc+pe}{mb+nd+pf}$.

3. The sum of n terms of an A. P. is 48, the first term is 20, and common difference -4. Find the number of terms and explain the double results.

4. Sum the following series.

$8-4+2-$&c. ... ... to 9 terms.
$\frac{a}{2}+\frac{2}{a}+\frac{8}{a^3}+$&c. ... ... to n terms.

5. Expand to 4 terms $\left(\frac{a^{\frac{1}{3}}x}{2} - \frac{2}{ax}\right)^{-2}$, and find the rth term.

6. Solve the following equations.

(1) $\frac{x-4a}{x-3a} + \frac{x-5a}{x-4a} = \frac{x+6a}{x-4a} + \frac{x+5a}{x-3a}$.

(2) $3x - \sqrt{(2x^2+6x+1)} = 1 - x^2$.

(3) $\left.\begin{aligned} xy + \frac{x}{y} = \frac{5}{3} \\ xy + \frac{y}{x} = \frac{5}{6} \end{aligned}\right\}$

7. A rides at the rate of 8 miles an hour; B walks at the rate of $3\frac{1}{2}$; B starts first, and after a certain interval A sets off to overtake him. When he had ridden 14 miles, his horse broke down, and he had to walk on, which he did at the rate of 4 miles an hour, overtaking B in 14 miles more. What start had B?

1867.

1. Reduce to their lowest terms.

$$\frac{a^2 - 8ax + 7x^2}{a^2 - 3ax - 28x^2} \text{ and } \frac{3x^3 + x^2 - 5x + 21}{6x^3 + 29x^2 + 26x - 21}.$$

2. If $a : b :: c : d$ prove that $ma + nc : pa + nc :: mb + nd : pb + nd$, and $a^4 - b^4 : (a-b)^4 :: c^4 - d^4 : (c-d)^4$.

3. Find the sum of the series.

$\frac{5}{6} + 1 + \frac{7}{6} + \&c.$ to 7 terms

$1 - 2 + 4 - \&c.$ to 8 terms

$1 + 2x + 3x^2 + 4x^3 + \&c.$ to n terms.

4. The common difference of an A. P. is 2, and the number of terms is equal to the second term. Find the first term in order that the sum may be 35.

5. Solve the following equations :—

(1) $10x - \frac{14x-9}{8x-3} = \frac{18-40x^2}{3-4x} - 9$.

(2) $\sqrt{\{x+\sqrt{(x^2-4)}\}}=\sqrt{\left(\frac{x+2}{2}\right)}+\sqrt{(2x-4)}.$

(3) $x^2-xy=8x+3$
$xy-y^2=8y-6.$

6. If in $(a+x)^n$ the 7th and 8th terms are in ratio as the 6th and 7th in $(a+x)^{n+1}$, find n.

Expand $(1+5x)^5$ and $(1-x)^{-2}$.

1868.

1. Simplify the expressions

(1) $\dfrac{a^{\frac{1}{3}}+3b^{\frac{1}{3}}}{a^{\frac{1}{3}}-3b^{\frac{1}{3}}}+\dfrac{a^{\frac{2}{3}}-3a^{\frac{1}{3}}b^{\frac{1}{3}}+9b^{\frac{2}{3}}}{a^{\frac{2}{3}}+3a^{\frac{1}{3}}b^{\frac{1}{3}}+9b^{\frac{2}{3}}}.$

(2) $\left(1-\dfrac{b^2}{a^2}\right)\left(1-\dfrac{ab-b^2}{a^2}\right)\dfrac{a^4}{a^3+b^3}\times\dfrac{a-b}{a^2+b^2}.$

2. Find the G. C. M. and L. C. M. of

$$x^3-x^2-4x+4 \text{ and } x^3-3x^2-4x+12.$$

3. If p be the difference between any given fraction and unity, and q the difference between its reciprocal and unity; then $pq=p-q$.

4. Solve the equations

$$\left.\begin{aligned}x+y&=7\\x^3+y^3&=91\end{aligned}\right\}\qquad\left.\begin{aligned}x+y+z&=18\\x^2+y^2+z^2&=110\\x(y+z)&=65\end{aligned}\right\}$$

5. Find the sum of n terms of the series

$$4+3+2+\ldots\ldots\ldots$$

$$a+\frac{2}{a}+\frac{4}{a^3}+\ldots\ldots$$

How many terms of the first series will amount to 4.

6. If $\frac{a}{b}=\frac{c}{d}$, show that $\dfrac{a+nb}{c+nd}=\dfrac{a-nb}{c-nd}$ and $\dfrac{a}{a+b}=\dfrac{ac-bc}{ac-bd}$.

7. How many different signals may be made with 5 flags; two of which are blue, one white, one red and one black.

8. Find the coefficient of x^{13} in the expansion of $(ax-x^2)^{10}$. Expand to 5 terms the series $\left(1-\frac{x}{2}\right)^{\frac{1}{2}}$, and find the r^{th} term.

1869.

1. If $x=\frac{\sqrt{2}+1}{\sqrt{2}-1}$ and $y=\frac{\sqrt{2}-1}{\sqrt{2}+1}$, find the value of x^2+xy+y^2.

2. Simplify

(1) $\frac{x^3-2x^2-5x+6}{x^3-4x^2+5x-6}$.

(2) $\{(x-y)^2+4xy\}^{\frac{1}{2}}\times\{(x+y)^2-4xy\}^{\frac{3}{2}}\times\left\{\frac{x^4-y^4}{x-y}+2xy(x+y)\right\}^{\frac{2}{3}}$.

3. If $a:b::b:c::c:d$, shew that $a:b::\sqrt[3]{a}:\sqrt[3]{d}$ and $(a+b)(c+d)=(b+c)^2$.

4. Solve the equations :—

(1) $ax^2-(a+b)x+a+b+c=0$.

(2) $2\sqrt{(x-1)}-\sqrt{(x+4)}=1$.

(3) $\left.\begin{array}{l} c(bx+ay)=axy \\ c(ax-by)=bxy \end{array}\right\}$

(4) $\left.\begin{array}{r} 27x+2y+3z=40 \\ x-y+z\quad=3 \\ 2x+4y-3z=12 \end{array}\right\}$

5. The first term of an A. P. is -2, the common difference 2; find the number of terms whose sum is 40.

Insert 4 Arithmetical means between 4 and -1.

6. Given the first term and the common ratio of a Geometric series, find the sum of n terms.

Sum the series :

(1) $a^n+a^{n-1}y+a^{n-2}y^2+\dots\dots\dots+ay^{n-1}+y^n$.

(2) $\frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^3}+\frac{4}{3^4}+\dots\dots\dots$ to n terms.

7. Find the middle term of $(a-b)^{20}$.

Expand $(a^2-x^2)^{-1}$ to 5 terms and find the rth term.

1870.

1. Reduce $\dfrac{3x^4-2x^2-8}{2x^4-3x^3+6x-8}$ to its lowest terms : and find the L. C. M. of $(x-2)^2$, x^3-8 and $2x^2-5x+2$.

2. Simplify $\dfrac{bc}{(a-b)(a-c)}(x-a)^2+\dfrac{ac}{(b-c)(b-a)}(x-b)^2$
$+\dfrac{ab}{(c-a)(c-b)}(x-c)^2$.

3. Solve the following equations.

(a) $\sqrt{(9x+1)}-1=3\sqrt{(-x)}$

(b) $\left.\begin{array}{l} x\sqrt{y}+y\sqrt{x}=-6 \\ x^{\frac{3}{2}}+y^{\frac{3}{2}}=26 \end{array}\right\}$

(c) $\left.\begin{array}{l} x^2+xy+y^2=28 \\ x-y-z=0 \\ \frac{x}{z}+\frac{z}{y}=\frac{3}{8}xz. \end{array}\right\}$

4. Given the first term and the common difference of a series of quantities in A. P., find the n^{th} term and the sum of n terms.

If the n^{th} term of an A. P. be a geometrical mean between the sum of the n terms and twice the common difference, show that the ratio of the first term to the common difference is $1\pm\sqrt{n}$.

5. Find the sum of

(a) $5+3+1-$ &c. to 20 terms.

(b) $\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2\sqrt{2}}+$.........to 10 terms.

(c) $3-1+\frac{1}{3}-$...........to n terms.

6. Find the 10th term of the expansion of $(a^2-ax)^{-2}$ and the middle term of $(2x-3y)^{10}$.

1871.

1. If $x+y+z=0$, show that

$$x\frac{y^3-z^3}{y-z}+y\frac{z^3-x^3}{z-x}+z\frac{x^3-y^3}{x-y}=0.$$

Find the G. C. M. of x^3-2x^2+3x-2 and x^2+x-2.

2. If $a : b :: c : d$, then $\frac{a^4+b^4+c^4+d^4}{a^{-4}+b^{-4}+c^{-4}+d^{-4}}=(abcd)^2$.

3. Solve the equations.

(i) $\frac{(x-1)(x-2)}{3}+\frac{2x-1}{5}=\frac{(2x-3)(x-4)}{6}$.

(ii) $\left.\begin{array}{l}\frac{1}{1-x+y}-\frac{1}{x+y-1}=\frac{2}{3}.\\ \frac{1}{x-y+1}-\frac{1}{1-x-y}=\frac{4}{3}.\end{array}\right\}$

(iii) $\left.\begin{array}{r}y-z=2\\ y(x+z)=65\\ z(x+y)=45\end{array}\right\}$

4. Given the first term and the common ratio of a geometrical progression, find the sum of n terms.

5. The common difference of an A. P. is -3, and the number of terms is equal to the third term. Find the first term when the sum is 18.

Sum the following series

$$\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}}+\frac{1}{2\sqrt{3}}-\text{\&c. to 10 terms.}$$

$$\frac{n+1}{n}-1+\frac{n}{n+1}-\frac{n^2}{(n+1)^2}+\text{\&c. to infinity.}$$

6. Expand by the Binomial Theorem $\left(\frac{x}{a}-\frac{a}{x}\right)^6$ and $(a-x)^{-3}$, the latter to 6 terms.

Find the middle term of $\left(\frac{y\sqrt{x}}{2}-\frac{2}{x\sqrt{y}}\right)^{16}$.

1872

1. Simplify (1) $\dfrac{(ay-bx)^2+(ax+by)^2}{\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{x}{y}+\frac{y}{x}\right)}$,

(2) $\dfrac{\left(x^3-1\right)\left(x-\frac{1}{x}\right)}{(x-1)^2(x+1)}-\dfrac{x^2+\frac{1}{x^3}}{x^2+\frac{1}{x^2}}-1$.

And find the G. C. M. of $x^3+3x^2-4x-12$ and $x^3+2x^2-9x-18$.

2. Solve the equations :—

$$\tfrac{2}{3}(5x+3\tfrac{1}{2})-\tfrac{1}{2}(6x-2\tfrac{1}{3})=6(14\tfrac{1}{3}-9x)$$

$$\left.\begin{aligned}6\sqrt{x}+5\sqrt{y}&=19\tfrac{2}{3}\\5\sqrt{x}+6\sqrt{y}&=17\end{aligned}\right\}$$

$$y^2z=2\sqrt{3},\ z^2x=3,\ x^2y=\sqrt{2}.$$

3. The first term and common difference of an arithmetic series being given; find the sum of n terms. Find the middle term of an arithmetic series, whose first term is 1, and last term 39, the number of terms being odd.

4. A geometric series consists of 6 terms, and the sum of the last three is 8 times the sum of the first three; find the common ratio.

Sum the series

$$13+11+9+\ldots\ldots\text{to 14 terms.}$$

$$2\sqrt{2}+\sqrt{2}+\frac{1}{\sqrt{2}}+\ldots\ldots\text{to 8 terms.}$$

5. Find the number of different signals that can be made with 6 flags, two of which are white, two black, and two red.

6. If n be a positive integer, prove that

$$1-n+\frac{n(n-1)}{1\cdot2}-\frac{n(n-1)(n-2)}{1\cdot2\cdot3}+\&c.=0.$$

Write down the first 5 terms of the expansion of $(a^3-x^3)^{\frac{1}{3}}$.

1873.

1. Simplify

(1) $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}\div\dfrac{\dfrac{a^2}{b^2}-\dfrac{b^2}{a^2}}{\left(\dfrac{1}{b}+\dfrac{1}{a}\right)^2}$; (2) $\dfrac{9x^3-3x^2y-6x^2+2xy}{6x^4-3xy^2-4x^3+2y^2}$.

(3) $\dfrac{\left(\dfrac{a^4-b^4}{a-b}+2ab(a+b)\right)^{\frac{2}{3}}}{\left\{(a-b)^2+4ab\right\}^{\frac{3}{2}}}$.

2. Solve the equations :—

(1) $\dfrac{x}{3}+\dfrac{3}{x}=\dfrac{x}{12}+\dfrac{12}{x}$. (2) $\left.\begin{array}{l}x^2+xy=15,\\ xy-y^2=2.\end{array}\right\}$

(3) $x^2yz=a,\ xy^2z=b,\ xyz^2=c.$

3. Find and explain the meaning of, the sum of the series $a+ar+ar^2+\ldots\ldots\ldots$ *ad inf.* r being a proper fraction. Sum to *six* terms and to infinity the series $3\frac{3}{8}+2\frac{1}{4}+1\frac{1}{2}+\ldots\ldots\ldots$

4. Out of a party of twelve how many sets of not more than five can be made up? How many times will any one person be chosen?

5. Expand $\left(2x^{\frac{2}{3}}-3y^{\frac{3}{2}}\right)^5$, and find the co-efficient of x^{10} in the expansion of $(1+x+x^2)^{-1}$.

Find by the binomial theorem, the cube root of 999 correctly to 8 places of decimals.

1874.

1. If $ax = \frac{2pq}{1+q^2}$, find the value of

$$\frac{\sqrt{\left(\frac{p}{a}+x\right)} + \sqrt{\left(\frac{p}{a}-x\right)}}{\sqrt{\left(\frac{p}{a}+x\right)} - \sqrt{\left(\frac{p}{a}-x\right)}}$$

2. Simplify (1) $\frac{x^2-bc}{(a-b)(a-c)} + \frac{x^2-ac}{(b-c)(b-a)} + \frac{x^2-ab}{(c-a)(c-b)}$.

(2) $\left\{\left(\frac{x^2}{y}+y-x\right) \div \left(\frac{x^2}{y^2}+\frac{y^2}{x^2}+1\right)\right\} \times \left(\frac{1}{y}+\frac{1}{x}+\frac{y}{x^2}\right)$.

3. Solve the equations : $\frac{x-1}{x^2+x} - \frac{x+1}{x^2-x} = \frac{1}{4-2x}$.

$$\left.\begin{array}{l} ax - by = a - b \\ a^2x + b^2y = a^2 - ab + b^2 \end{array}\right\}$$

$$\left.\begin{array}{l} x+y+z = xyz = 6 \\ xy+yz+zx = 11 \end{array}\right\}$$

4. Supposing the weight of an empty ship and the tonnage to vary as the square and as the cube of its length respectively : prove that, (w being the weight of a ship and its cargo, of length l, and w_1, w_2, similar weights corresponding to lengths l_1 & l_2)

$$\frac{w}{l^2}(l_1-l_2) + \frac{w_1}{l_1^2}(l_2-l) + \frac{w_2}{l_2^2}(l-l_1) = 0.$$

5. Find the sum of an Arithmetic series to n terms, and of a geometric series to infinity.

Find the number of terms in an Arithmetic series whose first term is 16, common difference 4 and sum 120, explaining the double result.

6. Find the greatest coefficient in the expansion of $(1+x)^{2n}$; and show that—

$$\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} = 1 + \frac{x}{a+x} + \frac{3}{2}\left(\frac{x}{a+x}\right)^2 + \frac{5}{2}\left(\frac{x}{a+x}\right)^3 + \&c.$$

1875.

1. Find the greatest common measure of the following quantities :—

(1) x^3-3x^2+4x-2 and x^3-2x+4.

(2) x^m-x^{-m} and $x^m+x^{-m}+2$.

2. Solve the equations—

(1) $\dfrac{mx}{mx+a}+\dfrac{nx}{nx+a}=2\dfrac{px}{px+a}$.

(2) $x^{\frac{1}{n}}+x^{-\frac{1}{n}}=a-2$.

(3) $yz=bc, \dfrac{x}{a}+\dfrac{y}{b}=1$ and $\dfrac{x}{a}+\dfrac{z}{c}=1$.

3. Give an algebraical solution of the proposition, Euclid Book II., Prop. 11. To divide a straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.

Interpret the two answers.

4. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$ prove that each of these ratios is equal to

$$\frac{x^2+y^2+z^2}{ax+by+cz}.$$

5. Find the number of terms of an arithmetical progression in which the first term, the common difference, and the sum of the terms is given. How many terms of the series 15+14+13+, &c., must be taken in order that the sum of the terms may be 120.

Explain the double answer in this case.

6. Find the number of combinations of n things taken r together. What is the number of different combinations of the letters in the word "Engineering" taken 5 together?

7. Assuming the binomial theorem for positive integral indices, prove it for fractional and negative indices.

Find approximately to 4 places of decimals the square root of 26.

1876.

1. Find the square of—

$$\frac{\sqrt{\{2ab+(a^2-b^2)\sqrt{(-1)}\}}+\sqrt{\{2ab-(a^2-b^2)\sqrt{(-1)}\}}}{a+b}$$

and the sum of

$$\frac{x}{x^2-1}+\frac{x^2+x-1}{x^3-x^2+x-1}+\frac{x^2-x-1}{x^3+x^2+x+1}-\frac{x^3}{x^4-1}.$$

2. Find the highest common divisor of $2x^3-10x^2+20x-16$ and $3x^3-12x^2+21x-18$, and the lowest expression in x which is divisible by each of the following—

$$6x^2-x-2,\ 21x^2-17x+2,\ 14x^2+5x-1.$$

3. Solve the equations :

$$(x+4)(x-3)+\sqrt{\{(x+3)(x-2)\}}=36 \quad \ldots \quad (1)$$

$$\left.\begin{aligned}\frac{x^2}{y}-\frac{y^2}{x}&=28\\ x-y&=8\end{aligned}\right\} \quad \ldots \quad (2)$$

$$\left.\begin{aligned}x^2+y^2+z^2&=2a^2\\ x+y+z&=2b\\ yz&=c^2\end{aligned}\right\} \quad \ldots \quad (3)$$

4. If from a series of n terms in arithmetical progression another series be formed, such that each term is an arithmetic mean between the consecutive terms of the first series, and if S and S′ be the sums of these series respectively, prove that $S : S' :: n : n-1$.

If a, b, c be in arithmetical progression, and x, y the geometric mean between a, b and b, c respectively, prove that b^2 is an arithmetic mean between x^2 and y^2.

5. Show that the number of combinations of n things three at a time is—

$$\frac{n(n-1)(n-2)}{1\cdot 2\,3}$$

A boat club consists of 15 members; in how many ways can a crew of 9 be chosen, (1) so as always to include, (2) so as always to exclude, a particular man.

6. In the expansion of a binomial when the index is a positive integer, prove that the coefficients of terms equidistant from each end are equal. Expand in ascending powers of x, as far as five terms,

$$\frac{1+2x-3x^2}{1-2x+3x^2}.$$

1877.

1. If $y=x-\frac{1}{x}$ express $x^8-\frac{1}{x^8}$ in terms of y.

Simplify

$$\frac{1}{\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right)}+\frac{1}{\left(1-\frac{c}{b}\right)\left(1-\frac{a}{b}\right)}+\frac{1}{\left(1-\frac{a}{c}\right)\left(1-\frac{b}{c}\right)};$$

and $\dfrac{\surd(12)}{(\surd 2-1)(\surd 6+\surd 3)}$.

2. Solve the equations :—

(1) $\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}=2a$.

(2) $\left.\begin{aligned}(x^2+y^2)(x-y)&=40\\ x^2+y^2-x+y&=18\end{aligned}\right\}$

3. If $ax+by=a+b$, and $a^2x^2+b^2y^2=a^2+b^2$, prove that $a^nx^n+b^ny^n=a^n+b^n$.

4. Find the sum of n terms of a geometric series.

Sum $1+\surd(10)+10+\cdots$ to n terms.

Write down the first three terms of the series whose p^{th} term is $p(3p-1)$, and find the sum of n terms.

5. Find the number of combinations of n things taken x together. Given ten white and ten black balls; in how many different ways is it possible to select from them a set of ten balls of which five shall be white and five black?

6. Expand $(a^2+x^2)^{-2}$ to six terms by the Binomial Theorem, and prove that

$$\left(\frac{a+x}{a-x}\right)^n=1+n\frac{2x}{a+x}+\frac{n(n+1)}{1\cdot 2}\left(\frac{2x}{a+x}\right)^2+\&c.$$

What is the sum of the co-efficients in the expansion of $(a+b+c)^7$?

1878.

1. Simplify the expression :—

$$\frac{yz}{(x-y)(x-z)}+\frac{zx}{(y-x)(y-z)}+\frac{xy}{(z-x)(z-y)},$$

and prove that

$$\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}=\left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)^2.$$

2. Solve the equations :—

(i) $$2\frac{a-bx}{ax-b}=\frac{a+bx}{ax+b}+1.$$

(ii) $$\left.\begin{aligned} x&=5\frac{1}{6}-\frac{1}{y}\\ y&=6\frac{1}{5}-\frac{1}{x}\end{aligned}\right\}$$ (iii) $$\left.\begin{aligned} x(y+z)&=a\\ y(x+z)&=b\\ z(x+y)&=c\end{aligned}\right\}$$

3. The first two terms of an arithmetical series are $1\frac{1}{2}$ and $2\frac{1}{3}$, find the 10th term. How many terms must be taken that the sum may be 171? Prove that

$$x^x+x^{x-1}(x-1)+x^{x-2}(x-1)^2+x^{x-3}(x-1)^3+\ldots\ldots$$
$$+x(x-1)^{x-1}+(x-1)^x=x^{x+1}-(x-1)^{x+1}.$$

4. Find the number of words which can be formed out of 8 letters taken all together, each word being such that 3 given letters are never separated.

5. What is meant by one quantity varying (1) directly, (2) inversely as another?

The value of diamonds varies as the square and of rubies as the cube of their weight, a diamond and a ruby weighing two carats each are of equal value, a diamond and a ruby each weighing three carats are together worth Rs 450, find the value of each.

6. Expand $(1-x)^{\frac{1}{n}}$ by the Binomial Theorem, and write down the $(x+1)^{\text{th}}$ term.

Prove that $\sqrt{2}=1+\frac{1}{2^2}+\frac{1\cdot 3}{\lfloor 2\cdot 2^4}+\frac{1\cdot 3\cdot 5}{\lfloor 3\cdot 2^6}+\&c.$

1879.

1. Break up the following expressions into all their factors.

$$x^8+y^8,\ x^{12}-y^{12} \text{ and } x^{16}+x^8y^8+y^{16}.$$

2. Solve the equations :—

(1) $\left(\dfrac{x-a}{x+a}\right)^{\frac{1}{4}}+\left(\dfrac{x+a}{x-a}\right)^{\frac{1}{4}}=\sqrt{3}\,;$

(2) $(x+b+c)\,(x+c+a)\,(x+a+b)$
$\qquad\qquad=(x+2a)(x+2b)(x+2c).$

(3) $x^2(y+z)=a^2 \qquad y^2(z+x)=b^2, \qquad z^2(x+y)=c^2.$

3. Sum the series

(1) $1+\dfrac{2}{5}+\dfrac{3}{5^2}+\dfrac{4}{5^3}+\&c.$ ad. inf.

(2) $1^2+2^2+3^2+4^2+\&c.$ to n terms.

4. Find the number of combinations of n things taken r together. How many combinations are possible out of 7 things?

5. State the Binomial Theorem.

Employ it to obtain the expansion of the following binomial expressions in powers of x to five terms :—

$$(a-x)^{\frac{4}{7}},\ \text{and}\ \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}.$$

Find the first negative term in the expansion of $\left(1+\dfrac{2x}{3}\right)^{\frac{21}{4}}$ and obtain the fifth root of 3225 correct to three places of decimals.

1880.

1. Prove $\dfrac{a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$

$$=(a+b)(b+c)(c+a)$$

$$\frac{x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)-4xyz}{x+y+z-xyz}$$

$$=1-xy-yz-zx.$$

2. Explain the principle of the solution of quadratic equations, and show that, in the equation $ax^2+bx+c=0$, the sum of the roots is $-\frac{b}{a}$, and the product of the roots is $\frac{c}{a}$.

Resolve into the product of two linear factors the quadratic expressions $7x^2-3x-160$, and $mnx^2+(n^2-m^2)x-mn$.

3. Solve the equations

(i.) $\frac{1}{a(b-x)}+\frac{1}{b(c-x)}=\frac{1}{ac-ax}$.

(ii.) $x^2+y(x+1)=4$.
$y^2+x(y+1)=2$.

(iii.) $ax+cy+bz=cx+by+az=bx+ay+cz$
$=a^3+b^3+c^3-3abc$.

4. Insert n Harmonic Means between a and b.

Prove that if a^2, b^2, c^2, be in Arithmetic Progression, then $b+c$, $c+a$, $a+b$, will be in Harmonical Progression.

5. Find the sum of the coefficients of the terms in the expansion of $(1+x)^n$ by the Binomial Theorem, where n is any positive integer. Show also that the sum of the coefficients of the odd terms is 2^{n-1}.

Find the coefficient of x^{2n-1} in the expansion of

$$\left(x-\frac{1}{x}\right)^{4n+1}.$$

6. Define a logarithm, and prove that

$$\log_b x=\frac{\log_a x}{\log_a b}.$$

Find the logarithm of $800\sqrt[3]{(50)}$ to the base $2\sqrt{5}$.

1881.

1. Reduce to its lowest term

$$\frac{x^4+5x^3+5x^2-5x-6}{x^4-5x^2+4}.$$

2. Solve the equations,

(1) $(x-2)(x-4)(x-6)(x-8)=9$:

(2) $x-2y+z=0$, $x+y+z=xy$, $xyz=105$.

3. Form an equation with real coefficients, two of whose roots shall be $\frac{\sqrt{(-3)}\pm 1}{2}$.

4. Find the greatest and least values of the expression $2x^2-6x-7$ for real values of x.

5. Prove that

$$(y+z)(z+x)(x+y)-x^3-y^3-z^3=4yz+(y+z-x)(z+x-y)(x+y-z).$$

6. How many different permutations can be made of all the letters of the word "engineering"? In how many of these will the three e's stand together, and in how many will they stand first?

7. Find the sum of all the coefficients in the expansion of $(1+x)^n$. Write down the first five terms of the expansions of

$$(3-2x)^7 \text{ and } (3-2x)^{-7}$$

8. Prove that $\frac{2}{\sqrt{5}}=1-\frac{1}{2^2}+\frac{1\cdot 3}{1\cdot 2}\cdot\frac{1}{2^6}-\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\cdot\frac{1}{2^9}+\&c.$

Hence determine the value correct to three places of decimals and verify the result.

9. Sum to n terms the series—

(1) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\&c.$

(2) $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+4\cdot 5\cdot 6+\&c.$

1882.

1. Find the roots of the equation $ax^2+2bx+c=0$.

If α and β be its roots then $a\gamma^2+2b\gamma+c=a(\gamma-\alpha)(\gamma-\beta)$.

(a). If $4xy=4ab=(x-a)(y-b)$, find x and y.

2. If $lx+my+nz=0$, $l_1x+m_1y+n_1z=0$

and $x^2+y^2+z^2=d^2$, find the value of x.

3. Sum the infinite series $1+2x+3x^2+4x^3+\&c.$, where $x<1$.

4. If the daily profit made by a building contractor vary directly as the sq. root of no. of masons he daily employs, the quality of the bricks and the no. of hours of labour remaining

constant from day to day, but he finds that his profit varies directly as the cube root of the no. of hours of daily labour if the no. of masons and the quality of the bricks remain the same from day to day, also he finds that if no. of masons and the no. of hours remain unchanged from day to day then his profit varies inversely as the price paid for the bricks. If on one day his profit be 420 Rupees when he employs 196 masons working $3\frac{3}{8}$ hours each, the price of bricks being Rs. 9 per 1000, you are, required to find the price of the bricks used on a second day when his profit is ~~Rs~~180, the no. of masons being 81, and the no. of hours being 8.

5. If the cube of the series $1+x+x^2+x^3+\&c$ to ∞ be expanded in a series of ascending powers of x, write down the coefficient of x^{11}.

6. Find the value of $1\cdot 2+2\cdot 3+3\cdot 4+4\cdot 5+\ldots\ldots n(n+1)$.

7. If $P=c^2(ax+b)^2\{3a(cx+d)-c(ax+b)\}$

and $Q=a^2(cx+d)^2\{3c(ax+b)-a(cx+d)\}$

find the value of the cube root of $(P-Q)$.

8. If $A=(b-c)(a-d)$, $B=(c-a)(b-d)$, $C=(a-b)(c-d)$, find the simplest value of $A^3+B^3+C^3-3ABC$.

1883. (F. C. E.)

1. Simplify $\dfrac{3abc}{bc+ca-ab}-\dfrac{\dfrac{a-1}{a}+\dfrac{b-1}{b}+\dfrac{c-1}{c}}{\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}}$: and show

that $a^4+b^4+c^4-(a^3+b^3+c^3)(a+b+c)$

$$+(a^2+b^2+c^2)(bc+ca+ab)=abc(a+b+c).$$

2. Show that a quadratic equation cannot have more than two roots. If m and n be the roots of the equation

$$x^2-(a+ab+b)x+ab(a+b)=0,$$

solve the equation $x^2-mx+n=0$ in which m and n are so chosen that the equation is homogeneous.

3. Find the number of combinations of n things taken r at a time. In how many ways can 2 fives, 3 fours, and 4 three's be

thrown with 9 dice, each die being in the form of a cube, the faces of which are numbered from 1 to 6.

4. An article is sold at a loss of as much per cent as it is worth in pounds. Show that £25 is the highest price at which it can be sold.

5. Show that in the expansion of $(1+x)^n$ where n is a positive integer, the coefficient of the r^{th} term from the beginning is equal to the coefficient of the r^{th} term from the end.

If $(1+x)^n=p_0+p_1x+p_2x^2+\dots+p_{n-1}x^{n-1}+p_nx^n$, find the value of $p_0^2+p_1^2+p_2^2+p_3^2+\dots+p_n^2$.

6. Show how to expand a^x in a series of ascending powers of x.

Prove that $\log_e\dfrac{1+x}{1-x}=2\left(x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\dots\right)$.

1883.

1. Find the square root of—

$$(1)\quad x^4-x^3+\frac{x^2}{4}-4x+2+\frac{4}{x^2},\qquad (2)\quad 1+(1-x^2)^{-\frac{1}{2}}.$$

2. Solve the equations :

$$(1)\quad \cdot 5x+\frac{\cdot 45x-\cdot 75}{\cdot 6}=\frac{1\cdot 2}{\cdot 2}-\frac{\cdot 3x-\cdot 6}{\cdot 9};$$

$$(2)\quad x^2+3-\sqrt{(2x^2-3x+2)}=\frac{3}{2}(x+1);$$

$$(3)\quad \frac{n}{x}+\frac{m}{y}=m^2+n^2,\ \frac{m}{nx}+\frac{n}{my}=m+n.$$

3. If the amount of sail a ship can carry varies as lbd, while the cargo she can carry varies as $\dfrac{b^2d^2}{l}$, where l, b, d denote her length, breadth and depth respectively; and if her speed varies as the amount of sail and the square of the length directly, and as the breadth and depth inversely; compare the speed of two vessels fully loaded and under full sail, one carrying 1200 sq. yds. of sail and 3600 tons of cargo, the other 810 sq. yds. of sail and 1458 tons of cargo.

4. Show how to find the sum of a given number of terms in G. P. Sum the series :—

(1) $-\frac{2}{3}, \frac{1}{3}, -\frac{1}{6}, \ldots\ldots$to 9 terms.

(2) $a^2-x^2, a-x, \frac{a-x}{a+x}, \ldots\ldots$to infinity.

5. Show how to find the numerically greatest term in the expansion of $(1+x)^n$ where n is a positive fraction.

Write down the numerically greatest term in the expansion of $\left(x-\frac{1}{x}\right)^{2x+1}$, where $x=\frac{7}{11}$.

6. Prove that $\log_a b \times \log_b a = 1$.

Given $\log_{10}4 = \cdot6020600$, $\log_{10}1\cdot04 = \cdot0170333$, find $\log_{10}\cdot01625$.

7. A person walking in a fog meets one man and overtakes another, both walking at the same rate. If he walks 21 yards and 126 yards between the times of first seeing and passing such of these respectively, find how far he can see in the fog.

1884. (F. C. E.)

1. Solve the following equations—

(1) $\{(1+x)^2-ax\}^{\frac{1}{2}}+\{(1-x)^2+ax\}^{\frac{1}{2}}=x.$

(2) $x^2+xy+y^2=37,\ x^2+xz+z^2=28,$
$y^2+yz+z^2=19.$

2. Find the sums of the cubes of the first n natural numbers.

Sum the following :—$\left(\frac{a-b}{a+b}+1+\frac{a+b}{a-b}+\text{\&c. to } n \text{ terms}\right)$

If $n^2+(n+1)^2+(n+2)^2+$ &c. to 9 terms $=501$, find n.

3. If P be the continued product of n quantities in G. P., s their sum and s_1 the sum of their reciprocals, shew that $P^2=\left(\frac{s}{s_1}\right)^n$.

4. Find the number of permutations of n things taken r together. How many different combinations of three letters can be formed from the letters of the word "Mathematician"?

5. Write down the first six terms of the expansion of $\left(2-\frac{3}{5}x\right)^{-5}$ and the general term of the expansion of $(1-x)^{-1}$.

Find the coefficient of x^n in the expansion of $\frac{1+x+x^2}{(1-x)^2}$.

6. State and prove the Exponential Theorem.

7. Define a logarithm. State some of the advantages of the use of logarithms. Shew that corresponding small increments of a number and of its logarithm are proportional.

1884.

1. Simplify $a\left(\frac{a^2b}{3a^2-6ac+3c^2}\right)^{\frac{1}{2}}+\frac{bc(ab)^{\frac{1}{2}}}{a-c}$ and

$$\left\{\frac{x+y}{x}+\frac{x+y}{y}+\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{2}}}+\frac{x^{\frac{1}{2}}-y^{\frac{1}{2}}}{y^{\frac{1}{2}}}+\frac{1}{4}\right\}^{\frac{1}{2}}.$$

2. Explain the term Greatest Common Measure of two algebraical expressions. Prove the rule for finding it.

Find the G. C. M. of

$e^{2x}a^3+e^{2x}-a^3-1$ and $e^{2x}a^2+2e^xa^2-e^{2x}-2e^x+a^2-1$.

3. Solve the equations—

(1) $x^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}=\frac{na}{(a+x)^{\frac{1}{2}}}$; (2) $ax+1=\frac{2ax(x^2+a^2)^{\frac{1}{2}}}{a-(x^2+a^2)^{\frac{1}{2}}}$.

4. Solve the equations—

(1) $(a+x)^{\frac{1}{4}}+(a-x)^{\frac{1}{4}}=h$;

(2) $x^{\frac{3}{4}}+y^{\frac{3}{4}}=126$, $\quad x^{\frac{1}{4}}+y^{\frac{1}{4}}=6$.

5. Prove, if α and β are the roots of the quadratic equation $x^2+px+q=0$, $\alpha+\beta=-p$ and $\alpha\beta=q$.

Find an equation with real coefficients which shall have for roots $1+\sqrt{2}$ and $2+\sqrt{(-3)}$.

6. Find the number of permutations of m things taken n at a time.

7. Expand $(1+mx)^{-\frac{1}{m}}$ to six terms by the Binomial Theorem.

8. If $a, b, c,$ be the p^{th}, q^{th} and r^{th} terms of an A. P. show that $p(b-c)+q(c-a)+r(a-b)=0$.

1885.

1. Simplify $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{(b+c)(c+a)(a+b)}{abc}$;

and find the G. C. M. of x^3-4x^2+2x+3 and $2x^4-x^2-5x-3$.

2. Solve the equations :—

(i) $a(x^2+1)=x(a^2+1)$.

(ii) $\left.\begin{aligned}\sqrt{(x+y)} &= \sqrt{(x)}+1\\ x-y &=7.\end{aligned}\right\}$

3. Write down the n^{th} term and the last term of the series $1+4+7+10+$&c. continued to $2n+1$ terms, and shew that if S_1 and S_2 be the sums of the first n and the last n terms respectively, then $S_2-S_1=6S$, where S is the sum of the first n natural numbers 1, 2, 3, &c.

4. Insert five geometrical means between 512 and 729.

If one geometrical mean G and two arithmetical means p and q be inserted between two given quantities, shew that

$$G^2=(2p-q)(2q-p).$$

5. If the quantity of water which flows through pipes in a given time vary as the square of their diameters, and two vessels whose contents are in the ratio of 26 to 9 be filled by two pipes respectively in 2 and 13 minutes, compare the diameters of the pipes.

6. Find the number of combinations of n things taken r at a time, and determine for what value of r the number of combinations of $2n$ things taken r together is greatest.

Shew that for this value of r the number of combinations which contains a particular thing is equal to the number of combinations which excludes it.

7. Enunciate the binomial theorem, and prove that the sum of the co-efficients of the odd terms in the expansion of

$(1+x)^n$ is equal to the sum of the co-efficients of the even terms. Write down the r^{th} term of $(1-x^2)^{-\frac{1}{3}}$.

8. Define a logarithm. Prove that $\log_m x + \log_m y = \log_m xy$. Given $\log_{10} 3 = \cdot 4771213$, $\log_{10} 7 = \cdot 8450980$, find

$$\log_{10}(\cdot 063)^{\frac{1}{3}} \text{ and } \log_3 \sqrt{(7)}.$$

9. Prove that $$a^x = 1 + (\log_e a)x + \frac{1}{\lfloor 2}(\log_e a)^2 x^2 + \frac{1}{\lfloor 3}(\log_e a)^3 x^3 + \ldots\ldots$$

Higher Examination Questions.

1. Find the present value of an annuity for a givén number of years at compound interest.

If V and $\frac{3}{2}$ V be the present values of the same annuity for n and $2n$ years respectively, then the Annuity is double of the simple interest upon V for one year.

2. If $\dfrac{x}{a^2+x^2} = \dfrac{2y}{a^2+y^2} = \dfrac{4z}{a^2+z^2}$,

shew that $x(y^2-z^2)+2y(z^2-x^2)+4z(x^2-y^2)=0$.

3. If $\dfrac{x(y+z-x)}{\log x} = \dfrac{y(z+x-y)}{\log y} = \dfrac{z(x+y-z)}{\log z}$

shew that $y^z z^y = z^x x^z = x^y y^x$.

4. Assuming that x^{-m} means $\dfrac{1}{x^m}$ when m is positive and that $x^{\frac{b}{a}}$ means $\sqrt[a]{(x^b)}$; prove that $\left(x^{\frac{b}{a}} \times x^{-\frac{y}{n}}\right)^{-\frac{r}{c}}$

$= x^{\frac{ayr-bnr}{anc}}$

where a, b, c, y, n, and r are positive integers.

5. If α, β be the roots of the equation $x^2+px+q=0$, then $\alpha+\beta=-p$ and $\alpha\beta=q$.

If one root of the equation $x^2+px+q=0$ be a root of $x^2+ax+b=0$, its other root is a root of $x^2+(2p-a)x+p^2-ap+b=0$.

6. Insert n harmonic means between two given quantities a and b.

If in a harmonic progression, the square roots of the 2nd, 3rd and 4th terms be in A. P., so are also the squares of the 1st, 3rd and 5th terms.

7. A four-wheeled carriage travels round on a circular railway. The circumferences of the fore and hind wheels of the carriage and of the two circles of rails are proportionals to 6, 7, 7000, 7014. Find the number of revolutions made by each of the four wheels in a complete circuit.

8. If at any place on the Ganges, the tide runs at the rate of 3 miles an hour, the flood lasting 5 hours and 20 minutes and the ebb for 7 hours, so that the tide runs down one hour and forty minutes longer than it runs up, and if the flood changes at different places gradually, so as to commence an hour later every 20 miles up the stream, show that a steamer going 9 miles an hour in still water will run up the river from Saugor Roads to Calcutta, a distance of 155 miles in less than one flood, but will require in running down from Calcutta to Saugor more than two floods and two ebbs.

9. If $\dfrac{x_1}{y_1}=\dfrac{x_2}{y_2}=\dfrac{x_3}{y_3}=$&c. show that each of these values is equal to $\left(\dfrac{m_1x_1{}^p+m_2x_2{}^p+m_3x_3{}^p+\&c.}{m_1y_1{}^p+m_2y_2{}^p+m_3y_3{}^p+\&c.}\right)^{\frac{1}{p}}$.

10. Eliminate l, m and n from the equations

$$a^2l^3+b^2m^3+c^2n^3=a^2l+\beta^2m+r^2n.$$
$$al=bm=cn$$
$$l^2+m^2+n^2=1$$

11. Find the value of an annuity at Compound Interest commencing at the present time, and show that the present value of an annuity to commence p years hence and to continue q years is the sum that in p years, would amount to the present value of the said annuity to commence immediately and last q years.

12. Find the value of $\log_{10}\cdot001$, $\log_7 2401$ and $\log_2\cdot015625$.

13. Colebrook's Lilavati, Ch. III. Sec. II. §49—"Pretty girl with tremulous eyes, if thou knowest the correct method of inversion, tell me what is the number which multiplied by 3, and added to three-fourths of this product and divided by 7, and then reduced by subtraction of a third part of the quotient, and then multiplied into itself, and having 52 subtracted from the product and the square root of the remainder extracted and 8 added and the sum divided by 10, yields 2 ?"

14. Sum to n terms the series

$$1+2x+3x^2+4x^3+\ldots\ldots\ldots\ldots\text{to infinity.}$$

Find the co-efficient of x^r in the expansion of

$$(1+2x+3x^2+4x^3+\ldots\ldots\ldots\ldots\text{to infinity})^n.$$

15. Solve the following equations :—

(1) $4\sqrt{(x^2+x+4)}=5x^2-4x-17.$

(2) $x+y=5xy,\ x^3+y^3=35x^3y^3.$

16. If $x+y+z=0$, shew that

$$\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right)=9.$$

17. Shew that

$$\frac{1\times2}{2^2}+\frac{2\times3}{2^3}+\frac{3\times4}{2^4}+\&\text{c. to infinity}=4.$$

18. A pack of cards contains 52 cards, each of which is marked differently. In how many ways may the cards be arranged in four sets, each set containing 13 cards ?

19. Sum the series $1\times3+2\times3^2+3\times3^3+\&\text{c}\ldots+n3^n$?

20. Sum the series

$$\frac{1\times2}{3^2}+\frac{2\times3}{3^3}+\frac{3\times4}{3^4}+\frac{4\times5}{3^5}+\ldots\text{to infinity.}$$

21. For what value of x the expression

$x^4+8x^3+22x^2+23x+19$ will be a perfect square.

22. If the two quadratic equations $ax^2+2bx+c=0$ and $a_1x^2+2b_1x+c_1=0$ have a common root, prove that

$$(ac_1+a_1c-2bb_1)^2=4(ac-b^2)(a_1c_1-b_1^2).$$

(*a*) If the roots of these equations be α, β and α_1, β_1 respectively, find the equation which has $\alpha\alpha_1 + \beta\beta_1$ for a root.

23. If $P = (a-b)^2 + (b-c)^2 + (c-a)^2$, express
$(a-b)^2(a+b-2c)^2 + (b-c)^2(b+c-2a)^2 + (c-a)^2(c+a-2b)^2$
in terms of P.

24. If $\sqrt{\{(1-x)(1+y+y^2)\}} - \sqrt{\{(1-y)(1+x+x^2)\}}$
$= C(x-y)$ and $\sqrt{(1-x^3)} - \sqrt{(1-y^3)} = (x-y)\sqrt{(a-x-y)}$.
express C in terms of a.

25. How many words of four letters can be formed from the letters a, b, c, d, e, f, l, m, n, o, p, r, s, t, u, supposing that no two consonants may stand together with the exceptions of p and r, p and t, s and l, s and r, s and t; and that more than two consonants can in no case stand together.

26. Ram, Lakshman and Bharat went to visit a Rishi, and brought their wives with them. The Rishi knew the wives' names to be Urmila, Mandavi and Sita, but forgot which was the wife of each hero. They told the Rishi that they had given presents to Pundits, and that each of the six had rewarded as many Pundits as he or she had given gold mudras to each Pundit. Ram had rewarded 23 more Pundits than Urmila, and Lakshman had rewarded 11 Pundits more than Mandavi, likewise each hero had given away 63 gold mudras more than his wife. The Rishi having thought on what they said, dismissed them with his blessing, naming correctly the wife of each hero. From the conditions given, do you also find out the names of the wives.

27. Shew that

$$\frac{\lfloor 2}{\lfloor 4} + \frac{\lfloor 4}{\lfloor 6} + \frac{\lfloor 6}{\lfloor 8} + \&c. = \frac{1}{2} \cdot \frac{1}{2^2} + \frac{1}{3} \cdot \frac{1}{2^3} + \frac{1}{4} \cdot \frac{1}{2^4} + \&c.$$

28. Shew that

$$\log_e \frac{e}{2} = \frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + \frac{1}{6\cdot 7} + \text{.........to infinity.}$$

29. Shew that

$$1 + (n-1)2 + \frac{(n-3)(n-2)}{1\cdot 2}2^2 + \frac{(n-5)(n-4)(n-3)}{1\cdot 2\cdot 3}2^3 + \ldots\ldots$$
$$= \tfrac{1}{3}\{2^{n+1} + (-1)^n\}.$$

30. Shew that,

$$1-2(n-1)+2^2\frac{(n-2)(n-3)}{1\cdot 2}+2^3\frac{(n-5)(n-4)(n-3)}{1\cdot 2\cdot 3}+\dots\dots$$
$$=\tfrac{1}{3}\{2^{n+1}-(-1)^{n+1}\}.$$

31. Prove that

$$n^n-n(n-1)^n+\frac{n(n-1)}{\underline{|2}}(n-2)^n-\frac{n(n-1)(n-2)}{\underline{|3}}(n-3)^n+\dots=\underline{|n}.$$

Given $\log_e 3=1\cdot 09861$, find $\log_{10}3$.

32. Prove that the arithmetic mean of any number of positive quantities is never less than the geometric mean.

33. If p be a proper fraction, and a and b be positive quantities, shew that $(a+b)^p a^{1-p}$ is less than $a+pb$.

34. If $\dfrac{x(y+z-x)}{\log x}=\dfrac{y(z+x-y)}{\log y}=\dfrac{z(x+y-z)}{\log z}$ prove that

$$y^z x^y=z^x x^z=x^y y^x.$$

35. Prove that the coefficient of x^r in the expansion of $\dfrac{1}{1-3x+2x^2}$ is $2^{r+1}-1$.

36. Prove that if $\log(1-x+x^2)$ be expanded in ascending powers of x in form

$$a_0+a_1x+a_2x^2+\dots\dots\text{then}$$
$$a_0+a_3+a_6+\dots\dots\dots=\tfrac{2}{3}\log 2.$$

37. Explain clearly the method of finding the G. C. M. of

$15x^4+4ax^3+15a^2x^2+15a^3x-90a^4$ and

$10bx^4+40abx^3-10a^2bx^2-10a^3bx-30a^4b$.

38. Prove that

$$(x+y+z)\{x^2+y^2+z^2-(y-z)^2(z-x)^2-(x-y)^2\}$$
$$=8xyz+(y+z-x)(z+x-y)(x+y-z).$$

39. Apply the binomial theorem to extract the cube root of 1333 to 6 places of decimals.

40. If $P=\dfrac{2x\sqrt{\{(1-x^2)(1-k^2x^2)\}}}{1-k^2x^2}$ and $Q=\dfrac{1-2x^2+k^2x^4}{1-k^2x^2}$, find the simplest value of P^2+Q^2.

Madras University Examination Papers.

[I]

1. Find the G. C. M. of

$$x^6+x^{-2}-(x^{-6}+x^2) \text{ and } x^3+x-(x^{-3}+x^{-1}).$$

2. If $a+\sqrt{b}=x+\sqrt{y}$, prove that $a=x$ and $\sqrt{b}=\sqrt{y}$.

Extract the square root of $\frac{3}{4}\pm\sqrt{2}$.

3. Extract the square root of

$$9a^2-12ab+24ac-16bc+4b^2+16c^2.$$

4. Shew that the roots of the equation $x^2+px+q=0$ are real and unequal, real and equal, or impossible, according as p^2 is greater than, equal to, or less than $4q$.

5. Solve the following equations :—

(a) $\dfrac{10x+3}{7}+\dfrac{5x+6}{4}=3x.$

(b) $25x\{\sqrt{(1-x^2)}-x\}=3.$

(c) $x^3+y^3+3axy=b^3.$
$x+y=c.$

6. A and B start from opposite corners of a square and run round in the same direction. If B stops for a certain time at every corner he will be caught by A, when he is just commencing his $(n+1)^{\text{th}}$ circuit, but if B runs continuously and A stops the same time at each corner, A will be caught when he is just commencing his $(m+1)^{\text{th}}$ circuit. Shew that A's vel. : B's vel. as $4+n^{-1} : 4+m^{-1}$.

[II]

1. Reduce to its simplest form the expression

$$\frac{1-x}{1-\dfrac{1}{1-x}}-x+\frac{1}{x}.$$

2. Shew that every common multiple of two numbers is a multiple of their least common multiple.

3. Explain the general principle of algebraical indices and show from it that $a^{-m}=\frac{1}{a^m}$ and $a^0=1$.

4. Shew that if α and β be the roots of the equation $ax^2+bx+c=0$, the expression ax^2+bx+c may be written $a(x-\alpha)(x-\beta)$.

What is the value of c in the equation $3x^2+6x+c=0$ when one root is $\frac{2}{3}$ of the other?

5. Solve the equation

$$\left.\begin{aligned} y+\frac{\sqrt{y}}{x}&=\frac{42}{x} \\ \frac{x^2}{3}+\frac{x}{2\sqrt{y}}&=\frac{54}{y} \end{aligned}\right\}$$

6. A merchant sent to his agent a certain number of bags of coffee for sale, expecting them to realize £250, and, after the agents commission of 4 per cent. was paid, to yield a profit of 20 per cent. on the original cost. On the way 5 bags were lost, but the rest were sold at 10*s*. per bag above the estimated price. This raised the net profits £76 17*s*. 6*d*. per cent. although the agent now received a commission of 5 per cent. How many bags were there at first?

[III]

1. Simplify

(1) $$\frac{\{x-a+\sqrt{(x^2-b^2)}\}\{x+a-\sqrt{(x^2+b^2)}\}}{\{x-a-\sqrt{(x^2-b^2)}\}\times\{x+a+\sqrt{(x^2+b^2)}\}}$$

(2) $\dfrac{a-x}{a+x}$ divided to four terms.

2. Reduce to their simplest forms:—

(1) $$\frac{x^3-11x^2+39x-45}{3x^2-22x+39}.$$

(2) $$\frac{\sqrt{(1-x^2)}+\dfrac{x^2}{\sqrt{(1-x^2)}}}{(1-x^2)^{\frac{3}{2}}}.$$

3. Shew that in finding the G. C. M. of two quantities, you may at any stage of the process reject any factor which is not common to both the quantities.

4. What relation must exist between the constants of the equation $x^2+px+q=0$, in order that the sum of its roots may be equal to three times their difference?

5. Solve the equations

(1) $$\frac{x+\sqrt{(x^2-a^2)}}{x-\sqrt{(x^2-a^2)}}=\frac{x}{a}.$$

(2) $$(x^2+y^2)xy=30$$
$$x^4+y^4=82.$$

6. In a school consisting of 3 classes, there were in the 2nd class 5 per cent., and in the 3rd class 10 per cent more than in the 1st class. In an examination, each boy in the first class occupied thrice, and in the 2nd class twice, as much of the examiner's time as each boy in the 3rd. The examination then lasted 31 hours. In the next year the 1st class had doubled in number, but each boy only required $\frac{2}{3}$ of his former time, there were 10 boys more in the 2nd class, and each boy in the 3rd class occupied $\frac{1}{11}$ hour more than he did before. The examination now lasted 43 hours. How many boys were there in the school at first?

[IV]

1. Find the G. C. M. of $21x^2+26xy+8y^2$ and $27x^2+42xy+16y^2$.

2. Prove that $\sqrt{\{-5+\sqrt{(-11)}\}}+\sqrt{\{-5-\sqrt{(-11)}\}}=\sqrt{2}$.

3. If a, b be the roots of the equation $x^2-px+q=0$, then $p=a+b$, and $q=ab$.

4. Solve the following equations :—

(i) $$\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}=ax$$

(ii) $$\left.\begin{array}{l} x^2+y^2=10 \\ (x+1)(y+1)=8 \end{array}\right\}$$

(iii) $$\left.\begin{array}{l} x(y+z)=44 \\ y(x+z)=50 \\ z(x+y)=54 \end{array}\right\}$$

5. A person bought one horse for x£, and another for y£, he sold the first at a profit of x per cent. and the second at a loss of y per cent., and thus received $\frac{1}{8}$th as much again as he would have, had he sold the *first* at a profit of y per cent. and the second at a loss of x per cent. If he had bought x horses at x£ each, and sold them at x per cent. profit, and had bought y horses at y£ each and had sold them at x per cent. loss, he would have gained altogether £1520. What did he give for each horse?

[V]

1. Simplify

$$(a+b)\frac{a+b-c}{2ab}+(b+c)\frac{b+c-a}{2bc}+(a+c)\frac{a+c-b}{2ac}.$$

State how the value of the expression will be affected by changing a into ma.

2. Substitute $\frac{1}{2}(b+c)$ for x in the expression

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-a)(x-c)}{(b-a)(b-c)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}.$$

and simplify the result.

3. What value of x will make the expression $(x+a)^2+b^2$ have the least possible value?

Divide 12 into two parts such that the square of one part together with 3 times the square of the other part may be the least possible.

4. State the relations which subsist between the roots of a quadratic equation and the co-efficients of its several terms.

If a and β be the roots of the equation $an^2+bn+c=0$, show that the equation whose roots are $a+n\beta$ and $\beta+na$ is

$$a^2a^2+(n+1)abx+nb^2+(n-1)^2ac=0.$$

5. Solve the following equations

(1) $\sqrt{(a+x)}+\sqrt{(a-x)}=\sqrt{\dfrac{a^2+2ab}{b+\frac{1}{2}a}}.$

(2) $xy+\dfrac{z}{y}=3$

$xz=2$

$\frac{9}{8}xz^2=(z^2-2x)^2.$

6. A person starts from A to walk to B and after he has gone 16 miles another starts from B to A and walks at double the rate of the former. Thirty-two minutes after they meet the slower of the two halts for two hours, and then does the remainder of the distance in 4 hours and 48 minutes. The other proceeds without stopping to A and immediately returns to B which he reaches 4 hrs. 24 min. after the other. Find the distance between A and B, and the distance from B of the halting place.

[VI]

1. Find the G. C. M. of two algebraic quantities.

Ex. $3x^4+4x^3+26x^2+28x+35$,

and $x^5+4x^4+12x^2+34x^3+35x+42$.

2. Find the square root of

$$x^6+4x^5+10x^4+20x^3+25x^2+24x+16.$$

3. Shew that

$$[\sqrt{\{(a^2-1)(x^2-1)\}}-ax-1]\times[\sqrt{\{(a^2-1)(x^2-1)\}}-ax+1]$$
$$\times[\sqrt{\{(a^2-1)(x^2-1)\}}+ax-1]\times[\sqrt{\{(a-1)(x-1)\}}+ax+1]$$
$$=(x^2-a^2)^2.$$

4. Solve the equations

$$\text{(i)}\quad \left.\begin{aligned} x^2+6xy+17y^2&=33\\ 3xy+16y^2&=22\end{aligned}\right\}$$

$$\text{(ii)}\quad \left.\begin{aligned} x^4+x^2y^2+y^4&=21\\ x+y&=7\end{aligned}\right\}$$

5. Three cases of goods cost Rupees 4000, they are sold again at a profit of 2, 3, 4 per cent. respectively, and the whole profit is 3 per cent. on the total cost; if the first and second cases had been sold for 5 rupees more each, and the third for what it had cost, the profit would have been 2 per cent.; what was the cost of each case?

[VII]

1. Find a value of x which will make $9x^4+12x^3+10x^2+8x+2$ a perfect square.

2. Shew that when $n+1$ figures of a square root have been obtained by the ordinary method n more may be obtained by simple division supposing $2n+1$ to be the whole number.

3. Shew that ax^2+bx+c and a never differ in sign unless the roots of $ax^2+bx+c=0$ are possible and different and x is taken so as to be between them.

4. Solve the following equations

(a) $x^4+2x^3+2x^2+x=\frac{1}{2}$.

(b) $xy-y^2+2x-y=2, \qquad x-y^2=1$.

5. The relative value of two sorts of mixed metals consisting of gold and silver is as 11 to 17. If the proportion of gold to silver in each had been doubled, their relative value would have been as 7 to 11. The value of gold to that of silver being as 13 to 1, find the proportion of gold to silver in each of the mixed metals.

[VIII]

1. If x^2+nx+1 and x^3+nx^2+1 have a common factor, determine n; and then shew that the L. C. M. of these quantities is $x^4-4x^3+2x^2+x-1$.

2. If lines be drawn through every two n points in a plane, prove that the number of triangles formed is $\frac{1}{48}$ $n(n-1)(n-2)(n^3-13n+20)$.

3. (a) Solve the equation

$$\sqrt{(x+3a)}+\sqrt{(x+3b)}+\sqrt{(x+3c)}=0.$$

(b) If $ax+by+cz=0$,

$a^2x+b^2y+c^2z=0$.

and $\dfrac{x}{b+c-a}+\dfrac{y}{a+c-b}+\dfrac{z}{a+b-c}=0$,

prove that $bc(b-c)^3+ac(c-a)^3+ab(a-b)^3=0$.

4. If the n^{th} term of a series in G. P. be $\dfrac{1}{2^n}\left(1-\dfrac{1}{2^n}\right)^2$ find the sum of the series to infinity.

5. Shew that the coefficient of x^4 in the expansion of

$\left(\frac{1+x}{1-x}\right)^n$ is $\frac{2}{3}n^2(n^2+2)$.

[IX]

1. Solve the equations :—

(1) $(x+4)(x+1)-5\sqrt{\{x(x+5)\}}=10$.

(2) $$\left.\begin{aligned} x+y+m(x-y)&=a \\ z-x-m(z+x)&=b \\ (1-m)^2yz&=ab \end{aligned}\right\}$$

2. If a number contain n digits, how many digits will be contained respectively in its second and third powers?

If x and a be integers containing respectively n and $2n+2$ digits, shew that $\frac{x^2}{a}+10^{n+1}\frac{x^3}{a^2}$ is less than $\frac{1}{5}$.

3. Sum the series

$$a^n+a^{n-2}+a^{n-4}+\&c.\ldots\ldots+a^{-n},$$

and write down the middle term when n is even, and the two middle terms when n is odd.

If bc, ac, ab are in harmonical progression

then $a\left(\frac{1}{a}+\frac{1}{c}\right)$, $b\left(\frac{1}{a}+\frac{1}{c}\right)$ and $c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in $A.\ P.$

4. Find the number of permutations of n things taken r at a time.

Find how many different numbers each beginning and ending with an even digit can be made with the digits 1, 2, 3,...9, all the digits occuring once and only once in each number.

5. Write down the coefficient of the $(r+1)^{\text{th}}$ term in the expansion of $(1+x)^{n+2}$ and shew that it is equal to the coefficient of the $(r+1)^{\text{th}}$ term from the end.

Expand (1) $\dfrac{x^{\frac{1}{3}}}{\sqrt[3]{(x+2x^2)}}$

$$(2)\quad \frac{(1-x)^{\frac{3}{4}}+\dfrac{1}{\sqrt{(1-x)}}}{2+x}$$

in ascending powers of x as far as x^2 inclusive.

6. Observing that

$$\frac{a-\dfrac{1}{a}}{(1-ax)\left(1-\dfrac{x}{a}\right)}=\frac{a}{1-ax}-\frac{\dfrac{1}{a}}{1-\dfrac{x}{a}}$$

find the sum of the series in question (3) by obtaining the coefficient of x^n on both sides of the identity.

[X]

1. If $x^2+y^2+z^2=1=a^2+b^2+c^2$, shew that

$$(1-ax-by-cz)(1+ax+by+cz)=(az-cx)^2+(bx-ay)^2 +(cy-bz)^2.$$

Solve the equation

$$\left.\begin{aligned}\frac{1}{x+1}+\frac{1}{1-y}&=\frac{7}{12}\\ x-y&=xy-13\end{aligned}\right\}$$

2. Determine the condition that the roots of a quadratic equation may be (1) rational (2) irrational (3) equal and of the same sign.

If $ax^2+bx+c=0$ have its roots equal and of the same sign, show that $a^2x^2+b(a+b)x+c(a+2b)=0$ has one the same, and find the other.

3. Show how to extract the square root of a Binomial Surd of the form $a+\sqrt{b}$.

Show that

$$\frac{1}{\sqrt{\{11-2\sqrt{(30)}\}}}-\frac{3}{\sqrt{\{7-2\sqrt{(10)}\}}}-\frac{4}{\sqrt{\{8+4\sqrt{3}\}}}=0.$$

4. Find the sum of the series

$$a+2a^2x+3a^3x^2+4a^4x^3+\ldots\ldots$$

(1) to n terms

(2) to infinity

If $\dfrac{x}{c+b-a}=\dfrac{y}{a+c-b}=\dfrac{z}{b+a-c}$.

Show that $(b-c)x+(c-a)y+(a-b)z=0$.

5. If ${}_nc_r$ denote the number of combinations of n things taken r together, show without the aid of formula that

$${}_nc_r=\frac{n}{r}\,{}_{n-1}c_{r-1}.$$

6. Find the middle term of $(1+x)^{2n}$. Show that when x is any positive integer except unity, $3^{3x}-26x-1$ is a multiple of 676.

[XI]

1. (*a*) Shew that $a^n-(-1)^nb^n$ is divisible by $a+b$.

(*b*) Thence deduce that $49^n+16n-1$ is divisible by 64.

2. (*a*) Find when ax^2+bx+c has not the same sign as a.

(*b*) If $x^2+2ax+4y^2-3a^2=0$, determine the limits between which x must lie for real values of y, and y for real values of x.

3. (*a*) Solve $x+y=3$, $y+z=5$, $x^3+y^3+z^3=36$.

(*b*) If the roots of $\sqrt{(ax+b)}-\sqrt{(bx+a)}=\sqrt{(a+b)}$ be equal, shew that $\dfrac{a}{b}$ must be one of the three cube roots of -1.

4. Find the formula for the number of combinations of n things taken r at a time, without assuming the formula for permutations.

5. Find the number of ways in which fourteen coins, not distinguishable from one another, may be divided among three persons so that no one may have less than three coins.

6. (*a*) Prove the Binomial Theorem for a negative integral index, assuming its truth for a positive integral index.

(*b*) Find the coefficient of x^{n-2} in the expansion of

$$\frac{x^3-3x+4}{(1-x)^8}$$

7. If p_n, $q_n\sqrt{(-1)}$ represent respectively the real and imaginary parts in the expansion of $\{1+\sqrt{(-3)}\}^n$, shew that $p_{n-1}q_n-q_{n-1}p_n=4^{n-1}\sqrt{3}$.

[XII]

1. Solve the equation

$$\frac{1}{2x^2-5x+3}+\frac{1}{3x^2-5x+2}=6x^2-23x+21.$$

2. If a and c have unlike signs, for what values of x has ax^2+bx+c a different sign from a.

If x is real, $\frac{(x-a-2b)(x-b-2a)}{(x-3a)(x-3b)}$ cannot lie between 1 and $\frac{1}{9}$.

3. Insert n harmonic means between a and b.

If the p^{th} term of a series in *H. P.* is q and the q^{th} term p, show that the pq^{th} term is 1.

4. Find the number of combinations of n things taken r at a time.

In how many ways can a pack of 52 cards be distributed equally among four persons so that,—(*a*) each may have ace, king, queen and knave of the *same suit*,—(*b*) each may have ace, king, queen and knave *all of different suits.*

5. Find the *numerically* greatest term in the expansion of $(1+x)^n$ taking n to be a positive fraction and x to lie between *one and zero.*

$$\text{Ex.}\quad\left(1+\frac{5}{11}\right)^{12\frac{3}{5}}.$$

6. Show that the coefficient of x^{11} in the expansion of

$$\sqrt{\left(\frac{(1+x)^5}{(1-x)^5}\right)}$$ in ascending powers of x is $\frac{7.9.11.23}{8\lfloor 4}$.

[XIII]

1. If $x+yz=\{(1-a^2)(1-y^2)(1-z^2)\}^{\frac{1}{2}}$

$$y+zx=\{(1-b^2)(1-z^2)(1-x^2)\}^{\frac{1}{2}}$$

and $z+xy=\{(1-c^2)(1-x^2)(1-y^2)\}^{\frac{1}{2}}$,

shew that $\dfrac{x^2-y^2}{a^2-b^2}=\dfrac{y^2-z^2}{b^2-c^2}=\dfrac{z^2-x^2}{c^2-a^2}$

2. If α and β are the unequal real roots of the quadratic equation $ax^2+bx+c=o$, the expression ax^2+bx+c has the same sign as a for all real values of x not lying between α and β.

For what values of x, in terms of α, β and r, would the expression $(a+bx+cx^2)+(2ax+bx^2)\,r+ax^2r^2$ have the same sign as a?

3. Given the sum of an arithmetical series, the first term and the common difference, find the number of terms and interpret the result when one of the values arrived at is a negative integer.

Shew, without applying the formula for the sum of the squares of the first n natural numbers, that

$$a^2+(a+b)^2+(a+2b)^2\ldots\ldots\ldots+(a+nb)^2=(n+1)a^2+n(n+1)ab$$
$$+\frac{n(n+1)(2n+1)b^2}{6}.$$

4. Find the number of permutations of n things all together, when they are not all different.

If X_r denote the number of permutations of x different things, taken r together, shew that $(M+N)_r$

$$=M_r+r.M_{r-1}\cdot N_1+\frac{r(r-1)}{1\cdot 2}\cdot M_{r-2}N_2+\ldots\ldots+r.M_1N_{r-1}+N_r.$$

5. Assuming the Binomial Theorem to be true for a positive integral exponent, prove it when the exponent is a negative integer.

Shew that $\sqrt[3]{\dfrac{10}{7}}=1+\dfrac{1}{10}+\dfrac{1\cdot 4}{10\cdot 20}+\dfrac{1\cdot 4\cdot 7}{10\cdot 20\cdot 30}\ldots\ldots ad.\ inf.$

[XIV]

1. Multiply $a^2+2b^2+9c^2-3ab+6ac-9bc$ by $a+2b-3c$ and divide the result by $a-b+3c$.

2. Simplify the fraction

$$\frac{(a^4-b^4)^2+2a^6b^2+5a^4b^4+2a^2b^6}{(a^2+ab+b^2)^2(a^2-ab+b^2)^2}.$$

3. Find the value of

$$\sqrt[5]{[(x-2)\sqrt{(x-2)}-2\{\sqrt[2]{(11x^2)}-x+2\sqrt{(x-2)}\}]}.$$

when $x=11$.

4. Find the square roots of

(1) $x^4-4x^3+6x^2-4x+1$.

(2) $x^2+4x+10+\dfrac{12}{x}+\dfrac{9}{x^2}$.

5. Solve the equations

(1) $\dfrac{x+3}{8}-\dfrac{x-3}{10}=\dfrac{x+5}{6}-\dfrac{x-7}{3}$.

(2) $\frac{1}{3}(\frac{x}{2}-2)-2(x-30)=\frac{1}{7}(x-6)-7$.

(3) $(a^2-b^2)x-(a^2-ab+c^2)y$

$$=a(a-2b)-\frac{bc^2}{a-b}.$$

$$\frac{x}{a}+\frac{y}{b}=\frac{2a}{a^2-b^2}.$$

6. A, B, C, D are four Railway Stations. From B to C is $2\frac{2}{3}$ miles more, and from C to D $5\frac{2}{7}$ miles less than from A to B. A train starts from A, and travels at the rate of 14 miles an hour. At B an accident happens to the engine, which causes a delay of 6 hours. After this the train proceeds to C at half speed. There another delay of $\frac{1}{2}$ an hour occurs, and then the train moves on to D at a speed further diminished by 1 mile an hour. A man starts from A at the same time as the train, and travels straight across country to D, a distance of 58 miles. Including stoppages he averages 3 miles an hour, and reaches D just with train. What is the distance by Rail from A to D?

[XV]

1. Solve the equation

$$(x+3)^2-2(x^2+3)=2x(x+1)^2.$$

2. Investigate the conditions under which the roots of the equation $ax^2+bx+c=0$, are real and equal, real and unequal, and imaginary.

3. (*a*) Form the equation of which the roots are

$$\tfrac{1}{2}\{(1+\sqrt{(3)}+\sqrt{(2\sqrt{3})}\} \& \tfrac{1}{2}\{(1+\sqrt{3}-\sqrt{(2\sqrt{3})}\}.$$

(*b*) Shew that the expression

$$\frac{(x^2-4)(x^2+3x+2)(x^2-x-2)+10}{x^2+5x+7}$$

is positive for all real values of x.

4. Find the sum to n terms of the series in G. P, whose first term is a, and the common ratio r.

(*a*) The first term of a Geometric Series continued to infinity is 1, and any term is equal to the sum of all the succeeding terms. Find the series.

5. Assuming the formula for permutations, find the number of combinations of n things taken r at a time.

(*a*) 25 passengers arrive at a Railway Station and proceed to the neighbouring village. At the station, there are 2 coaches accommodating 4 each, and 3 carts accommodating 3 each. Find the number of ways in which they can proceed to the village, assuming (1) that the conveyances are always fully occupied, and (2) that the conveyances are all distinguishable from each other.

6. Find the sum of the coefficients of the terms in the expansion of $(1+x)^n$, when n is a positive integer.

(*a*) If the coefficients in order are p_0, p_1, p_2, p_3, &c., deduce

$$p_0+\frac{p_1}{2}+\frac{p_2}{3}+\frac{p_3}{4}+\&c.=\frac{2^{n+1}-1}{n+1}$$

[XVI]

1. Solve the following equations :—

(1) $\sqrt[12]{(x^5)} - \frac{1}{2}\frac{p^2-q^2}{p^2+q^2}\{\sqrt{x}+\sqrt[3]{(x)}\}=0.$

(2) $x(9-xy)=y(xy-36).$

$xy(3x+12y-xy)=108\ (x+y-3).$

2. Show that a quadratic equation has two, and only two roots.

(*a*) Give the conditions, in order that the roots of the equation $\frac{x+a}{x}.\ \frac{x+b}{x}.\ \frac{x+c}{x}=1$ may be *possible*.

3. Insert n Harmonical means between a and b.

(*a*) Given that—

$(x-y)pq+(z-x)pr+(y-z)qr=0.$

Show that p, q, and r are the x^{th}, y^{th}, and z^{th} terms respectively of a $H.P.$

4. Find for what value of r the number of combinations of n things taken r at a time is greatest.

5. There are 4 packs of cards, each pack containing 52 cards. Find the number of ways they may be distributed among 4 persons, so that each may have 52 cards, of which there are 4 kings, assuming that the cards are all distinguishable from each other, and that the order among the persons remains unchanged.

6. Assuming the Binomial Theorem for an integer, prove it for a fractional quantity.

(*a*) When n is a positive integer, show that the sum of the squares of the coefficients of the terms in the expansion of $(x+a)^n$ is equal to $\frac{\lfloor 2n}{\lfloor(n)^2}$.

[XVII]

1. If a men or b boys can dig m acres in n days, find the number of boys whose assistance will be required to enable $(a-p)$ men to dig $(m+p)$ acres in $(n-p)$ days.

2. To complete a work A requires m times as long a time as B and C together; B requires n times as long a time as A and C together; and C requires p tines as long a time as A and B together. Compare the times in which each would do it and prove that

$$\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1.$$

3. Expand $\dfrac{1}{\sqrt[5]{(a^5-x^5)}}$ to five terms

and find the $(5+r)^{\text{th}}$ term.

4. Find the n^{th} term and the sum of n terms of the series

$$2+4+\frac{14}{3}+\frac{44}{9}+\frac{134}{27}.$$

5. How many different sums can be made with a guinea, a half-guinea, a crown, a half-crown, a shilling and six pence?

6. Find the square root of

$$x^m+\tfrac{1}{4}\sqrt[\frac{np}{2}]{(b^p x^{2n})}-\sqrt[4]{b}\,\sqrt[2p]{(x^{np+4})}.$$

[XVIII]

1. Divide $1+2x$ by $1-3x$ to 5 terms in the quotient.

2. Find the G. C. M. of

$$5x^3+2x^2-15x-6 \text{ and } -7x^3+4x^2+21x-12.$$

3. Find the value of x, y, z in the equations :—

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=12,\ \frac{y}{2}+\frac{z}{3}-\frac{x}{6}=8,\ \frac{x}{2}+\frac{z}{3}=10.$$

4. How many combinations can be made of 7 things 1, 2, 3, 4, 5, 6, 7 together respectively?

5. What is the logarithm of a number?

6. The sum of 700 Rupees was divided among 4 persons, whose shares were in geometrical progression; and the difference between the greatest and least was to the difference between the means as 37 to 12. What were their respective shares?

[XIX]

1. Give the square root of $42+3\sqrt{(174\frac{2}{5})}$.

2. Divide $2\sqrt{(-7)}$ by $-3\sqrt{(-5)}$.

3. Solve the following equations :—

(i) $\dfrac{1}{a^2-x^2}-a=\dfrac{ax}{a-x}+\dfrac{a}{a+x}$.

(ii) $x^2+x^{-2}=(x-x^{-1})^2+x$.

(iii) $\left.\begin{array}{l} x^2+y=11 \\ x^2+xy=15 \end{array}\right\}$

(iv) $\left.\begin{array}{r} (x^2+y^2)(x-y)=51 \\ x^2+y^2+x=20+y \end{array}\right\}$

4. Required two numbers such that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

5. Required the value in a series, giving 5 terms of $\dfrac{c^2}{(c^2-x)^2}$.

6. If m, n are any two numbers, g, their geometric mean, a_1, b_1, a_2, b_2, the arithmetic and harmonic means between m, g, and g, n, respectively, prove that $a_1 \; b_2=g^2=a_2b_1$.

7. Required the present value of a perpetual annuity of £55, allowing 3 per cent. compound interest.

8. Define the term logarithms, and state wherein the Napierian system, differs from the one now in general use.

Determine the base of the Napierian system

9. Given $\log 9 = \cdot 954242$

$\log 5 = \cdot 698970$

find the logarithms of $33\frac{1}{3}$, 6, 75.

[XX]

1. Express as a single fraction

$$\frac{a+x\sqrt{(-1)}}{a-x\sqrt{(-1)}}+\frac{a-x\sqrt{(-1)}}{a+x\sqrt{(-1)}}.$$

and shew that

$$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+1}=\sqrt{2}-\sqrt{3}+\sqrt{6}-2.$$

2. Find the values of x, y and z, from the equations:—

$$\left.\begin{array}{l}\frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=4z\\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=38\\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62\end{array}\right\}$$

3. Solve $x^3-2x=4$.

4. What are variations? How many variations can be made out of 6 things taking 5 together? What are combinations? How many combinations can be made of 6 things taking 5 together?

5. Find the 8th term and the sum to 8 terms of

(1) $1+\frac{6}{7}+\frac{5}{7}+\&c.$

(2) $2-4+8-\&c.$

6. Given $\log 2=\cdot30103$, calculate $\log 25$.

If $5^{x-1}=400$, find x.

[XXI]

1. Simplify $\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\times\left\{1+\dfrac{b^2+c^2-a^2}{2bc}\right\}$

and find the L. C. M. of

$x^2-(a+b)x+ab$, $x^2-(b+c)x+bc$, and $x^2-(a+c)x+ac$.

2. Solve the equations

(i) $\sqrt{\left(x-\frac{1}{x}\right)}+\sqrt{\left(1-\frac{1}{x}\right)}=x$

(ii) $\left.\begin{aligned} x^4 - x^2 + y^4 - y^2 &= 84 \\ x^2 + x^2y^2 + y^2 &= 49 \end{aligned}\right\}$

3. Sum the following series.

(1) $1{\cdot}2 + 2{\cdot}3 + 3{\cdot}4 + 4{\cdot}5 + \ldots\ldots$ to n terms.

(2) $\dfrac{\sqrt{2}+1}{\sqrt{2}-1} + \dfrac{1}{2-\sqrt{2}} + \frac{1}{2} + \ldots\ldots$ to infinity.

4. Expand by the Binomial Theorem to 4 terms;

$$(a^2 - ax)^{-\frac{3}{10}}.$$

5. A sets off from Madras to Bangalore, and B at the same time from Bangalore to Madras, and they travel uniformly; A reaches Bangalore 16 hours, and B reaches Madras 36 hours, after they have met on the road. Find in what time each has performed the journey.

[XXII]

1. Solve the equations

(i) $\dfrac{x+3}{x+2} + \dfrac{x-3}{x-2} = \dfrac{2x-2}{x-1}.$

(ii) $\left.\begin{aligned} x^2 + y^2 &= 7 + xy \\ x^3 + y^3 &= 6xy - 1 \end{aligned}\right\}$

2. A person has £ 1800 which he divides into two portions and lends at different rates of interest, so that the two portions produce equal returns. If the first portion had been lent at the second rate of interest it would have produced £ 50, and if the second portion had been lent at the first rate of interest, it would have produced £ 32. Find the rates of interest.

3. Explain generally the theory of Logarithms. Distinguish between Napierian and Natural Logarithms, and show how one can be converted into the other.

4. Find the square root of

$$7 - \sqrt{(10)} - \sqrt{(14)} - \sqrt{(35)}.$$

and expand by the Binomial theorem to 5 terms $(\sqrt[4]{a} + \sqrt[4]{x})^{-5}$.

5. How many different sums can be made up of a Rupee an eight-anna piece, a four-anna piece, a half-anna, and a pie.

[XXIII]

1. An odd number of Arithmetical means is inserted between -6 and 54 such that the last mean is double the middle mean. Find the number of means.

2. If in Geometrical Progression the x^{th}, y^{th}, and z^{th} terms are in Arithmetical Progression, prove that the sums of the series to x, y, and z terms are also in Arithmetical Progression.

3. Prove the formula for the combinations of n things taken r together, without assuming the formula for Permutations.

(*a.*) In how many ways can x boys and y girls be arranged in a straight line ($y < x$) so that no two girls shall stand together.

4. Find the coefficient of x^n in the expansion of

$$\frac{1+2x+3x^2}{(1-x)^3}$$

5. A person has P pounds which he puts out at interest, and he spends each year Q pounds *more* than the interest of P, and after a certain number of years he has nothing left. If he had spent each year Q pounds *less* than the interest on P, prove that at the end of the same time his property would have amounted to $2P$.

[XXIV]

1. Sum to n terms $1^3+2^3+3^3+$ &c.

(*a*) Sum to n terms $1\cdot 2^2+3\cdot 4^2+5\cdot 6^2+$ &c.

2. In Harmonical Progression, if the p^{th} term $=qr$, and the q^{th} term $=pr$, prove that the r^{th} term $=pq$.

3. Find the greatest numerical term of $(1+x)^{-n}$.

Ex. $(1+\frac{1}{2})^{-n}$, n being an integer.

4. If $p_r=\dfrac{3\cdot 5\cdot 7\ldots(2r+1)}{2\cdot 4\cdot 6\cdot 8\ldots 2r}$ and $q_r=\dfrac{1\cdot 3\cdot 5\ldots\ldots(2r-1)}{2\cdot 4\cdot 6\ldots\ldots 2r}$.

Prove that $p_r+p_{r-1}q_1+p_{r-2}q_2+\ldots\ldots+q_r=r$.

5. Find the sum of the first $2n+1$ coefficients in the expansion $\dfrac{1-x+x^2}{(1-x)^3(1+x)^2}$.

6. Find the amount of P£ in n years at compound interest, interest being paid q times a year, and each payment being $\dfrac{r}{q}$.

[XXV]

1. If $\sqrt{(a+\sqrt{b})}=\sqrt{x}+\sqrt{y}$, prove that $\sqrt{(a-\sqrt{b})}$

$$=\sqrt{x}-\sqrt{y}.$$

(*a*) Find the square root of $4+\sqrt{15}$.

2. The first term of a series in Arithmetical Progression is 12, the common difference 4, and the sum to n terms 132. Find n, and give a meaning to the negative value of n.

3. Sum to n terms, and to infinity

$$1+2x+3x^2+4x^3+\&c.$$

4. Assuming the Binomial Theorem to be true for any positive quantity, prove that it is true when the exponent is any negative quantity.

(*a*). Prove that

$$(1-2x)^{2n}=(1-x)^{2n}-2nx(1-x)^{2n-1}+\frac{2n(2n-1)}{1\cdot 2}x^2(1-x)^{2n-2}-\&c.$$

5. Find the number of homogeneous products of n things of r dimensions.

(*a*).—There are three dice, each die having six faces marked 1, 2, 3, 4, 5, 6. If the three dice are thrown together, how many different throws can be made.

6. Find the Present Value of an annuity for n years at Compound Interest.

(*a*).—What is the Present Value when the annuity lasts for ever?

[XXVI]

1. Resolve $3x^2-14xy+5x-5y^2+7y-2$ into factors.

2. Shew how to insert n harmonic means between two given numbers a and b.

An Arithmetical Progression and an Harmonical Progression have each the same first term a, the same last term l, and the same number of terms n. Prove that the product of the $(r+1)^{\text{th}}$ term of one series, and the $(n-r)^{\text{th}}$ of the other, is independent of r.

3. A contractor undertook to excavate a water reservoir at 2*s.* 6*d.* a cubic yard. The sides were vertical, and the length of one side of the base was 2 yards less than eight times the depth of the reservoir, also the length of the other side of the base was 5 yards less than six times the depth of the reservoir. There were 178 yards in the base more than in the four sides. Find the cost of excavation.

4. Prove that the total number of ways of taking n things is 2^n-1.

Expand $(1+x)^4(1-x^2)^{-3}$ in a series of ascending powers of x. Find the sum of the first n coefficients.

5. The simple interest on a certain sum of money for a certain time is £7, and the discount for the same time at the same rate of simple interest is £6. What is the sum of money?

[XXVII]

1. The product of four numbers in Arithmetical Progression is 4480, and their common difference is 6; find the numbers.

2. Derive the expression for the sum of an infinite number of terms of a series in Geometrical Progression; and sum to infinity the series—

$$\frac{2}{3}+\frac{5}{3^2}+\frac{2}{3^3}+\frac{5}{3^4}+\ldots\ldots$$

3. Expand $(a-b)^{\frac{1}{3}}$ by the Binomial Theorem; and use the result to obtain the cube root of 25 correct to five places of decimals.

4. Deduce the relation between logarithms of the same number to different bases.

Given $\text{Log}_\epsilon 9 = 2{\cdot}197225$.

$\text{Log}_\epsilon 10 = 2{\cdot}302585$.

find log 9 to base 10.

[XXVIII]

1. Divide $a + b^2 + c^3 - 3\sqrt[3]{(ab^2c^3)}$ by $a^{\frac{1}{3}} + b^{\frac{2}{3}} + c$.

2. Solve the following equations

(i) $x + y + z = 0$

$(a+b)x - (a+c)y + (b+c)z = 0$.

$abx - acy + bcz = 1$

(ii) $x^2 + y^2 = 2a^2$

$x + y : x - y :: m : n$.

3. The diagonal of a cube is a foot longer than each of the sides, find the solid content.

4. Prove that the sum of all the combinations that can be made of n things taken 1, 2, 3,.........n together is $2^n - 1$.

5. Find the sum of $\left(\frac{a^2}{x^2}\right)^{\frac{1}{3}} + \left(\frac{a}{x}\right)^{\frac{1}{3}} + \left(\frac{1}{a}\right)^{\frac{1}{3}} +$ &c. to infinity.

6. Expand by the Binomial Theorem $(1 - ax + bx^2)^{-\frac{1}{2}}$ to five terms.

7. Given log $2 = {\cdot}3010300$, log $3 = {\cdot}4771213$, find the logarithms of $3{\cdot}2$, $1\frac{1}{2}$, $\frac{2}{3}$, 15, ${\cdot}0054$, $14\frac{2}{5}$, $1{\cdot}8$ and $8{\cdot}1$.

[XXIX]

1. Find the L. C. M. of $a^3 + 2a^2b - ab^2 - 2b^3$

and $a^3 - 2a^2b - ab^2 + 2b^3$.

2. Sum the series

$$\frac{a}{r} + \frac{a+b}{r^2} + \frac{a+2b}{r^3} + \text{\&c. to } n \text{ terms.}$$

Ex. $\frac{1}{2}+\frac{3}{4}\div\frac{5}{8}+$&c.......

3. Find the number of combinations of n things taken r together.

Shew that for a given even value of n, the number of combinations is greatest when $r=\frac{1}{2}n$.

4. Find by the Binomial Theorem approximate values of $\sqrt{8}$ and $\sqrt[3]{9}$.

5. Solve the following equations

(i) $x^2+y^2=a^2$
$x+y=b$

(ii) $x+y+z=n$
$ax+by+cz=0$
$a^2x+b^2y+c^2z=0$

(iii) $20^x=100$, $\log 2=\cdot30103$.

6. It is found that in a given population the births in one year amount to $\frac{1}{m}$th of the whole, and the deaths to $\frac{1}{n}$th, in how many years will the population double itself.

[XXX]

1. In how many ways can you choose a pair out of twelve horses?

2. Form the equation whose roots are $6\pm\sqrt{(3-2\sqrt{2})}$. Are the roots real or imaginary?

3. The length of a room is 14 yards more than its breadth. Its area is 435 square yards; required its length and breadth.

4. In the returns of British Revenue for the last quarter, we find that the gross increase of revenue on the corresponding quarter of last year exceeded ten times the net increase by £93 more than a twelfth of the Post Office decrease.

There was an increase upon the "Crown Lands", a decrease of twenty-two times this amount upon "Stamps", and sixty times upon the "Post Office"; an increase upon "Taxes" equal in amount to the decrease upon the last items. The increase upon "Property Tax" exceeded that under the head of taxes by seven hundred times the increase on "Crown Lands", whilst the increase upon "Excise" fell short of that upon "Property Tax",

by seventy times the same quantity. Finally, the increase upon "Customs" exceeded by £313, thirty-one times the increase on "Crown Lands", whilst the decrease under the head "Miscellaneous" exceeded sixty-two times that under "stamps" by twice the increase on "Crown Lands" and £141.

What was the increase or decrease upon each item, the gross increase and the net increase?

5. Shew that $\frac{1}{1+x}=1-x+x^2-x^3+\&c.$

$$2^n=1+n+\frac{n(n-1)}{1\cdot2}+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}+\&c.$$

$$0=1-n+\frac{n(n-1)}{1\cdot2}-\frac{n(n-1)(n-2)}{1\cdot2\cdot3}+\&c.$$

[XXXI]

1. Simplify

$$\{[(x^2+y^2)^{\frac{1}{2}}]^6-(x^2-y^2)^3\}^2\times\{2y^2(3x^4-y^4)\}^3.$$

2. Find the G.C.M. and L.C.M. of $3a^3-11a^2x-17ax^2+4x^3$, and $7a^3-36a^2x+12ax^2-x^3$.

3. Solve the following equations

$$n^2m^2x^4+m^4=n^4+2m^3nx^2$$

$$x^3+x\sqrt{x}=72.$$

4. A tradesman turns over a third of his capital every three months, a sixth every eight months, an eighth every fourteen months, and the remaining three-eighths every two years. He makes a uniform profit of 25 per cent. and realizes an income of 600 per annum. What is his capital.

5. What are *similar surds*? show that the product of two dissimilar surds cannot be rational.

6. Find the square root of $2\frac{3}{4}-\sqrt{7}$.

7. If $a:b::c:d$ and $b:e::d:f$; then $a:e::c:f$.

8. State the Binomial Theorem. Assume its truth when the index is integral and prove it when the index is fractional.

[XXXII]

1. Multiply $14x^6+7x^5y+12x^3y^3-8x^2y^4+9xy^5-11y^6-4\dfrac{y^7}{x}$ by $(x+y)^4$.

2. Find the square root of $2\{6+\sqrt{(35)}\}$ in the form of a Binomial surd.

3. If the diameter of a circle be 14 inches long, and bisect a chord 4 inches long, shew that the segments of the diameter are represented in length by the two roots of the quadratic equation $x^2-14x+4=0$.

4. Give the number of ways, in which a ream of paper can be divided into half quires, stating your result in factors. Prove your rule.

5. A clerk falsifies his master's accounts by one pie the first day, three pies the second, five pies the third and so on. At the end of a year, by how much would he be able to defraud his master?

6. If in the above question we had written nine (instead of five) pies, within how many days would the same result have been obtained? Work by logarithms.

7. Solve the equation

$$\frac{m^2+n^2}{m^2-n^2}=x+\frac{mn}{(m^2-n^2)x}.$$

8. A certain number consists of three figures in A. P., reverse their order and you will have a number, whose excess above twice the original number is to its defect from three times that number in the duplicate ratio of 5 to 4, and also this defect is the square of the sum of the first and last figures. Required the original number.

[XXXIII]

1. Divide $a^{\frac{3}{5}}+b^{-\frac{3}{5}}$ by $a^{-\frac{1}{5}}+b^{\frac{1}{5}}$.

2. Find the L. C. M. of $a-x$, a^2-x^2, a^3-x^3.

3. Prove the truth of the following identities.

(1) $$\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}=\frac{x}{a-\sqrt{(a^2-x^2)}}.$$

(2) $$\frac{\sqrt{(a^2-x^2)}\sqrt[3]{(a+x)}}{\sqrt{(a-x)}\sqrt[3]{(a^2-x^2)}}=\sqrt[6]{\frac{(a+x)^3}{(a-x)^2}}.$$

4. Solve the following equations :—

(1) $\frac{1}{3}x-\dfrac{8x-1}{2}+3x=0.$

(2) $9x-3x^2+4\sqrt{(x^2-3x+5)}=0.$

(3) $\left.\begin{array}{l} x-y=1 \\ (x^2+xy+y^2)(x^3-y^3)=361. \end{array}\right\}$

(4) $x(y+z)=a,\ y(x+z)=b,\ z(x+y)=c.$

5. Shew that the equation $x^4+px^3+qx^2+rx+s=0$ may be solved as a quadratic if $r^2-p^2s=0$.

6. If A vary as B, then A is equal to B multiplied by some constant quantity. The volume of a right cone on a circular base varies directly as the radius of the base when the height is constant, and as square of the height when the radius of the base is constant. Find the height of such a cone which stands upon a base whose radius is 3 inches, and is made from two cones the radii of whose bases are 3 and 4 inches, and heights 4 and 6 inches respectively.

7. Find the sum of a series in A. P., having given the number of terms, the first term, and the common difference.

A ball is projected with such a velocity that it goes 20 feet in the first second ; in each second afterwards it goes a distance of 2 feet less than in the previous second. How far will it go in 9 seconds ?

8. A sum of money is left to two persons for their equal benifit. For the first it is to accumulate at compound interest, 5 per cent, for a certain time, and then to be given to the other for ever. Shew that the first person must keep it $14\frac{1}{5}$ years nearly. Given $\log 2 = \cdot 301030$, $\log 3 = \cdot 477121$, $\log 7 = \cdot 845098$.

9. Write down the $(r+1)^{\text{th}}$ term in the expansion of $(x+a)^{-n}$. From this find the greatest term in the expansion.

[XXXIV]

1. Resolve the following expressions into factors of the first degree $x(xy+x+1)-y(xy+y+1)$, x^2+xy+y^2.

2. Extract the square root of

$$x^2+11+\frac{1}{x^2}-6\left(x+\frac{1}{x}\right) \text{ and of } x^2-ax+a^2 \text{ to 4 terms.}$$

3. State and prove the rule for finding the L. C. M. of two algebraical quantities.

Find the L. C. M. of

$$a^2-3ab+ac-2bc \text{ and } a^2-b^2+2bc-c^2.$$

4. Solve the following equations :—

(a) $2x+y=4x^2,\ 3y-2x=y^2.$

(β) $\dfrac{x+m}{x-m}+m\dfrac{x+1}{x-1}+\dfrac{m+1}{m-1}=0.$

(γ) $x^3+6x^2-12x+8=0.$

5. Write down the r^{th} term of $(1+x)^n$ and of $(1-x)^{-2}$.

6. If ${}^nc_r$ represent the number of combinations of n things taken r together, prove that ${}^nc_r={}^nc_{n-r}$. Find the number of combinations of 31 things taken 28 together.

7. A person starting from a point A on a railway, after walking 10 miles meets a train which left B at the same time and which moves with the square of his velocity. Had he walked in the opposite direction he would have been overtaken after walking 15 miles. Find his rate of walking and the distance from A to B.

[XXXV]

1. If the equation $ax^2+bx+c=0$ have real roots, shew that for real values of x lying between those roots the expression ax^2+bx+c has an opposite sign to a and for all other values (the roots excepted) has the same sign as a.

Find x from the equation $x=\dfrac{b(3+4x)}{4-3x}$, and shew that in order that x may be real, b must not lie between 4 and $\frac{1}{4}$.

2. An odd number of arithmetical means is inserted between 7 and 43 such that the first mean is to the middle one as 2 to 5 ; find the means.

3. The permutations of n things taken r together is $n(n-1)(n-2)\ldots\ldots(n-r+1)$.

4. In the comparative examination the sixth classes of the three Provincial Schools are arranged in one list. Shew that, if the number in each class be n and if the boys of each school preserve relatively to each other a fixed order determined by a separate examination of each school there are

$$\frac{\lfloor 3n}{(\lfloor n)^3}$$

possible arrangements.

Show also that in $\sqrt{\dfrac{\lfloor 2n+1}{(\lfloor n)^2}}$ of these the boys of one particular school will stand together.

5. Write down the following quantities :—

(1) The coefficient of a^5b^7 in $(2a+3b)^{12}$.

(2) The coefficient of x^{3r+1} in $(x-x^{-2})^{2n+1}$.

6. If $f(m)$ denote the series $1+mx+\dfrac{m(m-1)}{1 \cdot 2}x^2+\&c.$, prove that $f(m) \cdot f(n)=f(m+n)$.

Hence prove the Binomial Theorem for a fractional index.

[XXXVI]

1. Simplify the expressions

$$\frac{\dfrac{a^2}{b-a}-a}{a-b}-\frac{\dfrac{a^2}{b-a}+a}{a+b}-\frac{\dfrac{2a^2}{b-a}}{a^2-b^2}.$$

and shew that

$$\frac{(n+1)^3-(n+1)}{3}ab+\left\{\frac{n(n+1)^2}{2}-\frac{n(n+1)(2n+1)}{6}\right\}(a^2+b^2)$$
$$=\frac{n(n+1)(n+2)}{6}(a+b)^2.$$

2. Solve the equation

$$x^4-2x^2(b^2+c^2)+(b^2-c^2)^2=0$$

and hence resolve the first member of the equation into four factors of the first degree.

3. Investigate the condition of equal roots in the equation

$$ax^2+bx+c=0.$$

Shew that, in order, that the values of x determined from the equations

$$ax^2+by^2=1$$
$$max+nby=1$$

may be equal, it is necessary that

$$m^2a+n^2b=1$$

and that, if this condition be satisfied,

then $x=m=yn$.

4. Find the harmonic mean between two quantities a and b, and shew that it is less than the geometric mean.

If A be the arithmetic, and H the harmonic mean between, a and b, shew that $\frac{a-A}{a-H}-\frac{b-A}{b-H}=\frac{A}{H}$.

5. Find the number of permutations of n things taken all together when p of them are of one sort, q of them another.

Shew that 504 numbers, each ending with two even digits can be formed with nine digits 1, 2, 3, 4......9 each digit occurring once and only once in each number, and the numbers being subject to the restriction that in reading them from left to right the odd digits shall be met with an ascending order of magnitude.

6. Shew that the coefficient of x^n in the expansion of $\frac{1+2x}{(1-2x)^2}$ is $(2n+1)2^n$.

[XXXVII]

1. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=$&c. prove that

$$\frac{a}{b}=\sqrt[n]{\left\{\frac{Aa^n+Bc^n+Ce^n+\ldots}{Ab^n+Bd^n+Cf^n+\ldots}\right\}}.$$

2. Solve the following equations :—

(i) $$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}+1&=\frac{11}{xy}, \\ \frac{1}{y}+\frac{1}{z}+1&=\frac{19}{yz}, \\ \frac{1}{z}+\frac{1}{x}+1&=\frac{14}{xz}. \end{aligned}\right\}$$

(ii) $$\left.\begin{aligned} &xy+y^2=3 \\ &x^4-y^4=6(x^2+3y^2)-27 \end{aligned}\right\}$$

3. Prove that $\sqrt{\{2-2\sqrt{(-3)}\}}=\sqrt{3}-\sqrt{(-1)}$.

4. Prove that the coefficient of x in the expansion of $(1+x)^n$ is n.

5. Find the number of combinations of n things taken r together.

If this number be represented by ${}^nc_r$, prove that

$$ {}^nc_{2r}+{}^nc_{2r-1}+{}^nc_{2r-2}+\ldots\ldots+{}^nc_2+{}^nc_1={}^{n+1}c_{2r}+{}^{n+1}c_{2r-2} $$
$$ +{}^{n+1}c_{2r-4}+\ldots\ldots+{}^{n+1}c_4+{}^{n+1}c_2. $$

6. In a series, any three consecutive terms beginning with an odd term are in A.P. and any three consecutive terms beginning with an even term are in H. P.; if a and $a+b$ be the first and second terms of such a series, prove that the sum of the series to 2^n terms $=\dfrac{2a+b}{2b}\cdot\dfrac{(a+2b)^n-a^n}{a^{n-1}}$.

7. Prove that $\log_e(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3\ldots\ldots$

[XXXVIII]

1. Shew that the expression

$$\frac{\{a+(n-1)b\}x^n}{x-1}+\frac{bx(1-x^{n-1})}{(x-1)^2}$$

will be increased by $(a+nb)x^n$, if n be increased by 1.

2. Investigate the circumstances under which the expression ax^2+bx+c will retain the same sign for all real values of x.

For what values of x will the expression

$$\frac{(x-3)(4x^2-4x+1)}{(x+2)(x^2-3x+3)} \text{ be positive ?}$$

3. Insert a given number of harmonical means between two given quantities.

If $2n$ arithmetical and the same number of harmonical means be inserted between a and b, and s and s' be the sums respectively of the two middle arithmetical and the two middle harmonical means show that $s' = \dfrac{(2n+1)^2 abs}{n(n+1)s^2+ab}$.

4. There are $3p$ things of which $2p$ are alike and the rest unlike; find the number of combinations of them taken $2p$ at a time. Find also the number of permutations taken $2p$ at a time.

5. If two consecutive terms in the expansion of $(x+a)^n$. [n an integer], are equal, these two terms are the greatest terms. Prove this and find these terms when $n=3m+2$ and $x+2a$.

In the expansion of $(x+a)^{199}$ whether is the 66^{th} or 68^{th} term the greater when $x=2a$?

6. Assuming $a^x = 1 + c_1x + c_2x^2 + c_3x^3 +$ &c., find c_2, c_3 &c. in terms of c_1.

What is the value of c_1?

[XXXIX]

1. If $a+b+c+d=0$ and $a^{-1}+b^{-1}+c^{-1}+d^{-1}=0$, prove that $a^3+b^3+c^3+d^3=0$.

2. State and prove the rule for finding the L. C. M. of two quantities.

Find the L. C. M. of x^4-4x-8, and x^3-2x-4.

3. Solve the equation

$$(1) \quad \frac{1+a^2x^2}{1-ax} - \frac{1+b^2x^2}{1-bx} = (a-b)x$$

$$(2) \quad \frac{b^2-x^2}{b^2-m^2} = \frac{x}{m} \cdot \frac{a-m}{a-x}.$$

4. Show how to find the sum of a series in A. P.

Find the sum of n terms of each of the following series :—

$$1+3+5+7+\&c.$$
$$1\cdot3+3\cdot5+5\cdot7+\&c.$$

5. Assuming the truth of the Binomial Theorem when the index is a positive integer, prove it when the index is a negative integer.

Find the coefficient of x^{r+1} in the expansion of

$$(1+ax+a^2x^2+a^3x^3+\&c.\text{ to infinity})^n.$$

6. If x differs very little from one, show that

$$\frac{ax^b-bx^a}{x^b-x^a}=\frac{1}{1-x}\text{ very nearly.}$$

[XL]

1. Find the G. C. M. of $3x^2-x-10$ and $2x^2-3x-2$, and the L. C. M. of x^4+x^2-2 and x^3-3x^2-x+3.

2. Solve the following equations :—

(1) $\dfrac{x^2+2x-3}{x^2+5x+6}-\dfrac{x}{x+2}=\dfrac{1}{2}$.

(2) $\dfrac{x+1}{x+2}=x^3$.

(3) $x+a\sqrt{y}=y+a\sqrt{x}=b^2$.

(4) $\left.\begin{aligned}4x(x+y)&=3y\\ y^2+2&=6x\end{aligned}\right\}$

3. Find two numbers, of which the product shall be 6 and the sum of their cubes 35.

4. Reduce $(32a^6-96a^5x)^{\frac{1}{5}}$.

Add $\sqrt{(72)}$ and $\sqrt{(128)}$.

Find the difference of $\sqrt{(27a^4x)}$ and $2\sqrt{(3a^2x^3)}$.

Multiply $2\sqrt[3]{(10)}$ by $4\sqrt[3]{(12)}$.

Divide $\sqrt{8}$ by $\sqrt[3]{(16)}$.

5. Find the sum of $2ab+ab+\frac{ab}{2}+\frac{ab}{4}+$&c. to infinity

Show how the sum of the G. P. a, ar, ar^2, ar^3...to n terms is obtained, and when r is less than unity, show the expression for the sum of the progression when n is infinitely great.

If a, b, c are in *H. P*, prove that

$a-\frac{b}{2}$, $\frac{b}{2}$, $c-\frac{b}{2}$ are in G. P.; and if the latter progression be 2, 10, 50, find a, b, c.

6. Expand by the Binomial Theorem to five terms.

$$(a^2+x^2)^{\frac{1}{2}} \text{ and } \frac{1}{\sqrt{(b^2+x^2)}}.$$

7. In what time will a sum of money be doubled, at $4\frac{1}{2}$ per cent. compound interest, supposing the interest to be due half-yearly.

8. A certain number of persons, represented by the numbers 1, 2, 3, 4 &c. may be arranged in 35 rows, three in each, so that no two persons shall be abreast twice. There are 7 rows of even numbers, and the other rows are each formed of two odd numbers, and one even number. Find how many persons are required for the above combination, and arrange them in the 35 rows.

[XLI]

1. Find the G. C. M. of x^5-4x^2-4x-8 and x^8-16, and resolve it into quadratic factors.

2. If $\phi(a)=\frac{(a-\alpha)(a-\beta)(a-\gamma)}{a}$, prove the identity

$$\frac{\phi(a)}{(a-b)(a-c)}+\frac{\phi(b)}{(b-c)(b-a)}+\frac{\phi(c)}{(c-a)(c-b)}=1-\frac{\alpha\beta\gamma}{abc}.$$

3. Solve the equations

(i) $\sqrt[3]{(x+a)}+\sqrt[3]{(x+b)}+\sqrt[3]{(x+c)}=0$

(ii) $(b+c)x+(c+a)y+(a+b)z=0$

$$bcx+acy+abz=0$$
$$ax+by+cz=a+b+c.$$

and from your solution of the first deduce the inequality $a+b+c>3\sqrt[3]{(abc)}$ when a, b, c are positive and not all equal.

4. Find the number of combinations of n things taken r together. If the solid angles of an icosahedron be joined in every possible way, how many lines will there be, exclusive of the edges of the figure?

5. Prove the Binomial Theorem for a positive integral index.

If $(1+x)^c$ be expanded and then arranged according to powers of x, what is the numerical coefficient of x^6?

6. Two rectangles of equal area have their sides in the ratios $a:\ b$, and $a^3:\ b^3$ respectively, and the perimeter of the latter exceeds that of the former by $2(a+b)$, show that the area of either is $\dfrac{a^3b^3}{(a-b)^4}$.

[XLII]

1. If $x+y+z=1$,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

and $ax+by+cz=a+b+c$, find the value of $a^2x+b^2y+c^2z$.

2. State and prove the rule for converting a recurring decimal into a vulgar fraction.

3. Prove that

$$2\cdot1^2+3\cdot2^2+4\cdot3^2+\&c.\ldots\ldots\text{to } n \text{ terms}$$
$$=\tfrac{1}{12}n(n+1)(n+2)(3n+1).$$

4. When $n+1$ figures of a square root have been obtained by the ordinary method, shew that n more may be obtained by division only.

5. The value of the expression a^3-3a^2+2 remains unaltered when a is increased by $2\sqrt{3}$. Determine that value.

6. Find the number of homogeneous products of r dimensions that can be formed out of n letters a, b, c......and their powers.

[XLIII]

1. Resolve $6x^2y^2-7xy^3-3y^4$ into elementary factors ; and simplify

$$\frac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}}\times\frac{x^2-y^2}{x^3+y^3}.$$

2. Solve the equations

(a) $\sqrt{(3x+4)}+\sqrt{(3x-5)}=9.$

(b) $\sqrt{(9x)}+\sqrt{x}=\sqrt{(15x+4)}.$

3. Find the n^{th} term and the sum to n terms of the series $1-3+5-7+\&c.$; and the 20^{th} term and the sum to 20 terms of $1-2+4-8+\&c.$

4. Expand $(2x^3-3y^2)^5$ by means of the Binomial Theorem.

5. Solve the equation $8x^3+16x-8=1$.

6. Define a logarithm. Reduce the negative logarithm $-(3{\cdot}1415926)$ to another of which the characteristic only shall be negative. Find the value of $\cdot9977$ and $\cdot8333$.

[XLIV]

1. Prove the rules for multiplying Algebraic quantities. Perform the following operations :—

(1) $\{-x+y\sqrt{(-1)}\}^3.$

(2) $\sqrt{\left\{(2e^x+e^{-y})(2e^y+e^{-x})+2\left(2e^{\frac{x+y}{2}}+\frac{1}{2}+e^{-\frac{x+y}{2}}\right)\right\}}.$

2. Define the L. C. M. of two quantities.

Every common multiple of two quantities is a multiple of their L. C. M. If the Algebraic expressions A. B. and their greatest common measure be the m^{th}, n^{th} and r^{th} degree in x respectively, no common multiple of A and B can be of less than the $(m+n-r)^{\text{th}}$ degree.

3. Explain the method of Mathematical Induction?

If there be n factors, each of which is the sum of two squares, show by the above method that their product may be expressed as the sum of two squares, and determine the number of ways in which this may be done.

4. Show without the aid of formula that the number of combinations of n things taken r together is the same as the number taken $n-r$ together.

Find the number of different combinations of 10 things 5 together of which 4 are alike.

5. Form the product

$$(1+a_1x)(1+a_2x)(1+a_3x)\ldots\ldots\ldots(1+a_nx)$$

and deduce the Binomial Theorem for a positive integral index.

If ${}^nC_r$ be the coefficient of x^r in $(1+x)^n$, show that

$${}^nC_1+\frac{\lfloor m}{\lfloor m-1}\,{}^nC_{r-1}+\frac{\lfloor m}{\lfloor 2\lfloor m-2}\,{}^nC_{r-2}+\ldots\ldots\ldots+{}^nC_{r-m}$$

$$=\frac{\lfloor m+n}{\lfloor r\lfloor m+n-r}.$$

6. Explain the "Principle of proportional parts" employed in finding the log of a given number from the tables.

Construct a table of proportional parts and use it

(1) in finding d when

$$\log\left(1+\frac{d}{n}\right)=\cdot0000175.$$

(2) in finding $\log\left(1+\frac{d}{n}\right)$ when $d=\cdot75$.

[XLV]

1. Prove the rule for finding the G. C. M. of two algebraical quantities.

Show that ax^3+bx^2+cx+d and dx^3+cx^2+bx+a will have a common divisor of the second degree if

$$\frac{abc-a^2b-b^2d+acd}{ac-bd}=\frac{ac^2-bcd-a^3+ad^2}{ab-cd}=\frac{d(ac-bd)}{a^2-d^2}.$$

2. If α and β be the roots of the equation $x^2+px+q=0$, then $\alpha+\beta=-p$ and $\alpha\beta=q$.

If one root of the equation $x^2+(c+d)x+q=0$, be a root of the equation $x^2+(c-d)x+b=0$, its other root is a root of the equation $x^2+(c-3d)x+(c+d)^2-(c^2-d^2)+b=0$.

3. If I buy n articles for 1 Rupee each, $n-1$ articles for Rs2 each, $n-2$ for Rs3 each, and so on up to one article for Rs n each, find the sum I shall have to pay.

4. If the 7th and 8th terms in $(a+b)^n$ are in the same ratio as the 6th and 7th in $(a+b)^{n+1}$, find n.

5. Find the present value of an annuity to commence at the end of p years, and to continue q years.

A person 30 years of age having Rs 20,000 wishes to purchase an annuity which is to begin when he attains the age of 50 : if his life be valued at 60 years, and interest be reckoned at 3 per cent., find the value of the annuity he may purchase, having given

$\log 3 = \cdot 4771213$	$\log 1\cdot 8061 = \cdot 2567418$
$\log 2 = \cdot 3010300$	$\log 2\cdot 4272 = \cdot 3851056$
$\log 1\cdot 03 = \cdot 0128372$	$\log 6\cdot 211 = \cdot 7931615$
	$\log 4\cdot 2348 = \cdot 6268329.$

[XLVI]

1. Find the limits between which

$$\frac{\left(x-\frac{a+nb}{n+1}\right)\left(x-\frac{b+na}{n+1}\right)}{(x-a)(x-b)}$$ cannot lie for real values of x.

2. If n harmonic means be inserted between p and q, shew that the sum of the rth and $(n-r+1)$th means is less than $p+q$.

3. Find the number of ways in which the letters a, b, c, d, e may be arranged, so that (1) a may always appear before, b, (2) a, b, c may not appear in the 1st, 2nd and 3rd places, respectively.

4. Show by mathematical induction that $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is divisible by 2^n, n being a positive integer.

5. Find the greatest term in the expansion of $(x+a)^n$, if n is a positive integer and no two consecutive terms of the expansion are equal.

If $x=3a$, which is greater term in the expansion $(x+a)^{100}$, the 49th or the 52nd; and with the same condition, which is the greatest term in the expansion $(x+a)^{198}$.

6. Assuming that $a^x = 1 + c_1 x + c_2 x^2 + \&c.$, prove that

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \&c.$$

Shew that the sum to infinity of the series whose $(r+1)^{\text{th}}$ term is $\dfrac{(r+1)^3}{\underline{|r}}$ is $15e$.

[XLVII]

1. Shew that the equation

$(b-x)^2-4(a-x)(c-x)=0$ has real roots whatever be the values of a, b, c.

Shew that the expression $(a-k)x^2+(b-k)x+(c-k)$ will have an invariable sign for all real values of x, if k have any value outside certain limits.

2. In the expansion of $(1+x)^n$ where n is a positive integer, determine the circumstances in which (1) the terms increase from beginning to end, (2) there are two consecutive terms equal.

In the expansion of $\left(1+\dfrac{1}{10}\right)^{\frac{222}{2}}$ find the first term that is less than the preceding term, and also the first term that is negative.

3. Find the first two terms in the expansion of

$$\frac{2+3x+(1-3x)^{\frac{1}{2}}}{1-\frac{1}{2}x+(4-x)^{\frac{1}{2}}}$$

according to ascending powers of x.

4. Assuming the formula

$$\log_e(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\&c.$$

prove that $\log_e(n+1)-\log_e n=2\left\{\dfrac{1}{2n+1}+\dfrac{1}{3}\,\dfrac{1}{(2n+1)^3}+\&c\right\}$ given $\log_e 3=1{\cdot}098612$, find $\log_e 10$ to 5 places of decimals.

[XLVIII]

1. Explain and illustrate the method of Mathematical Induction.

Shew that the sum of the products of n quantities taken three and three together $=\frac{1}{6}(s_1^3+2s_3-3s_1s_2)$, where s_1 is the

sum of the quantities, s_2 the sum of their squares, and s_3 of their cubes.

2. Find the *numerically* greatest term in the expansion of $(a+x)^n$ when n is negative.

3. Four men and two women have to sit at a round table; in how many different ways can they be arranged so that the women are not together? In how many such arrangements are three of the men together?

4. Expand $\left(1+1+\frac{1}{1.2}+\frac{1}{1.2.3}\ldots\ldots\right)^x$ in ascending powers of x.

Shew that $4e=1{\cdot}3+\frac{2{\cdot}4}{1{\cdot}2}+\frac{3{\cdot}5}{1{\cdot}2{\cdot}3}+\frac{4{\cdot}6}{1{\cdot}2{\cdot}3{\cdot}4}\ldots\ldots$ *ad inf.*

[XLIX]

1. Show how imaginary expressions may be introduced into Algebraical investigations. Prove that the modulus of the quotient of two imaginary expressions is the quotient of their moduli.

Solve the equation $\left(x-\frac{a^2}{x}\right)\left(x-\frac{\beta^2}{x}\right)\left(x-\frac{\gamma^2}{x}\right)$

$$=(x-a)(x-\beta)(x-\gamma),$$

where $4a^2\beta\gamma$ is positive and greater than $(a\beta+\gamma a-\beta\gamma)^2$.

2. Show how to insert any number of geometrical means between two given terms.

Construct an infinite geometrical series such that the sum of all the terms after any selected term is equal to that selected term.

3. Solve the equations :—

i. $\frac{x+b}{a}+\frac{x+a}{b}=\frac{a}{x+b}+\frac{b}{x+a}$.

ii. $\left(\frac{x^2-13x+31}{x^2-x-11}\right)^2+3\frac{x-3}{x+1}=0$.

4. Define the terms *logarithm* and *mantissa*. What is the *common* system of logarithm? Show the relation between the logarithms of the same number to different bases.

[L]

1. Solve the following equations :—

$$4y^2+3xy-x^2=0.$$

$$(y-a)(x-y)=(x+2a)(x+y)$$

2. Find for what relation between the coefficients and what values of x the expression m^2x^2+bx+c is negative.

If x be real, show that the expression $\frac{m^2}{1+x}-\frac{n^2}{1-x}$ can have any real value.

3. Find the sum of the cubes of the first n natural numbers.

Show that the sum of the cubes of any number of consecutive integers (*not* beginning with unity) is divisible by the sum of the integers.

4. Find the number of permutations of n things taken all together which are not all alike.

Show that 154 numbers less than 1000 and divisible by 5 can be formed with the ten digits, each digit not occurring more than once in each number.

5. Find the greatest term in the expansion of $(x+a)^n$, n being a positive integer.

Show that the coefficient of x^n in the expansion of $\frac{(1-2x)^n}{1-3x}$.

[LI]

1. Solve $(x+m)^{\frac{2}{5}}+(x-m)^{\frac{2}{5}}=\left(n+\frac{1}{n}\right)(x^2-m^2)^{\frac{1}{5}}$.

2. (*a*) If x is real, shew that ax^2+bx+c and a differ in sign only when the equation $ax^2+bx+c=0$ has real roots and x is taken to lie between them.

(*b*) If x is real, the value of $\frac{(x-1)(x+3)}{(x-2)(x+4)}$ does not lie between $\frac{4}{9}$ and 1.

3. If a, b, c, d, are proportionals, shew that

$\frac{ma+nb}{pa+qb}=\frac{mc+nd}{pc+qd}$; and that $\frac{a^2}{b}:\frac{c^2}{d}$ is inversely as $\frac{a}{b^2}:\frac{c}{d^2}$.

4. (*a*) Find the m^{th} term of a *H. P.* whose first term is a, whose last term is c and whose number of terms is n.

(*b*) Shew that the sum of n terms of a *G. P.* beginning with the p^{th} term is r^{p-q} times the sum of an equal number of terms of the same series beginning with the q^{th} term.

5. (*a*) Assuming the truth of the Binomial Theorem when the exponent is a positive integer, prove it for any positive exponent.

(*b*) Find the $(2n+1)^{th}$ term from the beginning in the expansion of $\left(x-\frac{1}{x}\right)^{3n}$.

[LII]

1. Solve the following equations :—

(*a*) $$\frac{2x^2}{x(15+x)-8x}+\frac{123+41\sqrt{x}}{\sqrt{x}(5-\sqrt{x})}=\frac{20\sqrt{x}+4x}{3-\sqrt{x}}$$

(*b*) $$\left.\begin{array}{l} x^2+3x+y=73-2xy \\ y^2+3y+x=44 \end{array}\right\}$$

2. Extract the square root of $p+q\sqrt{(-1)}$ and thence shew that any even root of this expression is of the form $P+Q\sqrt{(-1)}$.

Find the fourth roots of (-4).

3. Four pieces of white paper, on one of which a mark is made, are put into a hat; and five pieces of coloured paper, one of which also is marked, are put into another hat. In how many ways can four pieces, *viz.* two from each hat be drawn without the two marked pieces appearing simultaneously ?

4. Find the amount of Rs 5,000 lent at compound interest at 4 per cent. per annum for 10 years, one half of the annual interest being converted into principal every half year.

5. (*a*) If $x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}=a(x^2-y^2)$ and $xy-$

$$\sqrt{\{(1-x^2)(1-y^2)\}}=b(x^2+y^2-1),$$

show that $\frac{1}{a^2}+\frac{1}{b^2}=1$.

(b) Solve for x and y the simultaneous equations

$$(1+m^2)(x+y)=2m(1+xy)$$
$$(1+n^2)(x-y)=2n(1-xy)$$

6. Assuming the Binomial Theorem for a positive index, integral or fractional, prove it for a negative index. Show that the coefficient of x^n in the expansion of

$$\frac{1}{(1-x)(1-2x)(1-3x)} \text{ is } \tfrac{1}{2}(3^{n+2}-2^{n+3}+1).$$

[LIII]

1. Solve the equations

(a) $2x-x^2+\sqrt{(6x^2-12x+7)}=0$

(b) $x+y+z=\dfrac{a^2}{x}=\dfrac{b^2}{y}=\dfrac{c^2}{z}$.

(c) $49(x^2+a^2)(y^2+b^2)=50(xy+ab)^2$.
$(x^2-a^2)(y^2-b^2)=24(bx-ay)^2$.

2. Prove that the equations $x^2+19x-140=0$, and $7x^4-12x^3+87x^2-1392x+70=0$, have two common roots.

3. A field redoubt consists of a space enclosed by an earthen parapet which is lined by men and guns. Each gun requires 5 lineal yards of parapet, and the rest of the parapet is occupied by infantry at 1 man per yard. In addition, there is a reserve of infantry amounting to $\frac{2}{3}$ of the garrison. A width of 15 ft. inside the parapet is required for the fighting line; and the remaining internal space is the accommodation for men and stores. Each man of the garrison needs at least 15 sq. ft. of this space, and each gun 600 sq. ft. Find the side of the smallest square redoubt capable of containing three guns, and an infantry garrison, and find the number of the garrison.

4. A person who has a fixed life income of Rs 1000 per mensem resolves to set apart a portion of it to purchase a policy of insurance, so that, on his death, the sum paid by the Insurance Company may be sufficient, when invested at 4 per cent., to give his family an income equal to that which he will be enjoying at the time of his demise. If Rs 3·6 is the annual premium on a policy of Rs 100, find the amount he must devote annually to insurance.

5. Find the coefficient of x^m in the expansion of $\dfrac{1}{x^2-5x+6}$.

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Bombay University Examination Papers.

[I]

1. Solve the following :—

(i) $\frac{2}{3}x - \frac{1-\frac{1}{2}x}{4x} = \frac{x-1}{2} + \frac{x}{6} + \frac{7}{12}$.

(ii) $\left.\begin{aligned} x+2y+3z&=17 \\ 2x+3y+\ z&=12 \\ 3x+\ y+2z&=13 \end{aligned}\right\}$

(iii) $\left.\begin{aligned} x^4+y^4&=1+2xy+3x^2y^2 \\ x^8+y^8&=2y^2x+2y^2+x+1. \end{aligned}\right\}$

2. Two master-brick layers undertake to lay the foundation of a new court, each taking a part and beginning together. If they had worked together till the whole was finished, it would have taken only $\frac{4}{5}$ of the time it actually took to finish it ; and B would have done enough to occupy A three months, and A enough to occupy B twelve months, which is 36 yards more than A actually did. How many yards were there in all ?

3. Express in the form of the sum of two simple surds the roots of the following equations :—

(i) $x^4-2ax^2+b^2=0$

(ii) $4x^4-4(1+n^2)\,a^2x^2+n^2a^4=0$.

4. Given $\log \frac{1}{2} = \bar{1}\cdot6989700$; $\log \frac{1}{3} = \bar{1}\cdot5228787$; find the logs of $\sqrt{3}$, $\sqrt{2}$, $\frac{2}{3}\sqrt[3]{(\cdot05)}$ and $\frac{12}{25}\sqrt[4]{(1\cdot6)^3} \times \sqrt[3]{(21\cdot6)^4}$.

5. A country trebles its population in a century ; what is the increase per year per million : given

$$\log 3 = \cdot4771213$$
$$\log 101\cdot1 = 2\cdot0047512$$
$$\log 101\cdot11 = 2\cdot0047941.$$

6. Write down the general terms of

(i) $(a^2+x^2)^{-\frac{2}{3}}$ and (ii) $(a^2-x^2)^{\frac{2}{3}}$

7. If the m^{th} term of an $A.\ P.$ be n, and the n^{th} term m, how many terms must be taken so as to give the sum $\frac{1}{2}(m+n)(m+n-1)$? and what will be the last of them ?

8. Determine m and n in terms of a and b so that $\dfrac{ma+nb}{m+n}$ may be the arithmetic mean between m and n and the geometric mean between a and b.

[II]

1. What is meant by the modulus of a system of logarithms? What is the base of the ordinary or Brigg's system, and what the base of the natural or Napierian system of logarithms.?

2. Given $\log 2 = \cdot30103$ and $\log 3 = \cdot4771213$; find the value of $\sqrt[3]{\left\{\sqrt{\left(\dfrac{750\times24}{5\times4}\right)}+34\right\}}$, and find the logarithm of 768 and of 324.

3. Solve the equations

(i) $\left.\begin{aligned}\sqrt{y}-\sqrt{(7-x)}+\sqrt{(y-x)}=0\\ \sqrt{y}-\sqrt{(5-x)}-\sqrt{(y-x)}=0\end{aligned}\right\}$

(ii) $\left.\begin{aligned}\frac{1}{x}-\frac{1}{y}=\frac{1}{a}\\ lx+my=\frac{h}{k}xy\end{aligned}\right\}$

4. There are two numbers each consisting of three digits and whose sum falls short of 1000 by unity. If they be separated by a decimal point, the results obtained by putting one number first and then the other will be in the proportion of 1 : 6. Find the numbers.

5. Expand $(\frac{x}{3}-3y^2)^4$ by the binomial theorem.

6. Out of seven Hindus and eight Parsees five are to be selected which must always consist of 3 Parsees and 2 Hindus; in how many ways may the selection be made?

[III]

1. Given $\log 2 = \cdot30103$ and $\log 32 = 1\cdot50515$; find $\log \sqrt[3]{(\cdot625)}$, $\log \cdot0000025$, and the value of the expression

$$\sqrt{\frac{128\times2\cdot5\times80}{\cdot001\times500\times2}}.$$

2. Solve the equations.

$$\left.\begin{aligned} 2^x &= 3^y \\ 2^{y+1} &= 3^{x-1} \end{aligned}\right\}$$

3. Find the value of each of the following expressions.

(a) $$\frac{x^2y^2}{(z^2-x^2)(z^2-y^2)}+\frac{x^2z^2}{(y^2-x^2)(y^2-z^2)}+\frac{y^2z^2}{(x^2-y^2)(x^2-z^2)}.$$

(b) $$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}}\div\frac{\frac{1}{a}-\frac{1}{b}+\frac{1}{c}}{\frac{1}{a}-\frac{1}{b}-\frac{1}{c}}.$$

4. Extract the square root of

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{9}{4}.$$

5. Solve the equations

(a) $(3x-1)^2+(4x-2)^2=(5x-3)^2$

(b) $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$.

6. A man has a number of Rupees which he tries to arrange in the form of a square. On the first attempt he has 130 over. He then increases the side of the square by 3 Rupees and he has only 31 over. How many Rupees had he?

7. Two pieces of cloth are bought for Rs 48 8 as. One is three yards longer than the other, and each cost as many half Rupees the yard as it is yards in length. What are their lengths?

[IV]

1. Show some of the practical advantages of the common system of logarithms over other systems.

2. In what time will a sum of money double itself at 5 per cent per annum compound interest?

Given $\log 2 = \cdot 30103$

$\log 105 = 2\cdot 02119$.

3. Find the G. C. M. of $270a^2b^3x^2z$, $90a^3bx^3$, $432a^4b^2xy$, and the L. C. M. of bx^2-x-1 and $2x^2+3x-2$.

4. Extract the square root of

(i) $7+2\sqrt{(10)}$.

(ii) $5+\sqrt{(10)}-\sqrt{6}-\sqrt{(15)}$.

5. Solve the following

(i) $\frac{x}{a}+\frac{y}{b}=1$; $\frac{a}{x}+\frac{b}{y}=4$.

(ii) $x+y=4\sqrt{(xy)}$; $x-y=c\sqrt{\frac{x}{y}}$.

6. A person has rupees 1300 which he divides into two portions and lends at different rates of interest so that the two portions produce equal returns. If the first portion had been lent at the second rate of interest, it would have produced Rs 36 ; and if the second portion had been lent at the first rate of interest, it would have produced Rs 49. Find the rates of interest.

7. A person who saved every year half as much again as he saved the previous year had in seven years saved Rs 1029 8 as. How much did he save the first year ?

[V]

1. Simplify the following

(*a*) $2\sqrt[4]{(92{\cdot}351)}+\sqrt[3]{(95443{\cdot}993)}-4\sqrt{(167{\cdot}9616)}$

(*b*) $\sqrt[4]{(7-4\sqrt{3})}$.

2. What are common logarithms and what Napierian? What is the *modulus* of the common system ? Shew that the Napierian base is incommensurable.

3. If $3s=a+b+c$, prove that

$$(s-a)^4+(s-b)^4+(s-c)^4=2\{(s-b)^2(s-c)^2+(s-c)^2(s-a)^2 + (s-a)^2(s-b)^2\}.$$

4. Distinguish between a quadratic equation and quadratic expression.

Shew that whatever real value x may have, ax^2+bx+c and a never differ in sign except when the roots of the equation $ax^2+bx+c=0$ are possible and different, and x is taken so as to lie between them.

5. The sum, first term and common difference of an A. P. being given, find the number of terms, and explain why two values are generally obtained. How many terms of the series ·0053, ·0059, &c. will amount to ·0665?

6. Find the sum of the fourth powers of the first n natural numbers.

7. What do you mean by an infinite series in G. P.? Shew how to effect the summation of such a series, and hence find the value of a mixed circulating decimal.

8. Investigate the greatest term in the expansion of $(1+x)^n$, n being positive and integral.

9. Find the number of homogeneous products of r dimensions which can be formed out of n letters and their powers.

[VI]

1. Given log 2 = ·30103 and log 7 = ·845098; find the logarithms of 35, 49, ·00175 and ·01715. Find also of how many digits 2^{64} will consist.

2. Simplify $$\frac{\dfrac{y}{1+y}+\dfrac{1-y}{y}}{\dfrac{y}{1+y}-\dfrac{1-y}{y}}$$

and $\left\{x^{-\frac{3}{2}}y\left(xy^{-\frac{1}{2}}\right)\cdot\left(x^{-1}y\right)^{-\frac{2}{3}}\right\}^3$.

3. Solve the equations :—

$$\left.\begin{array}{r}\sqrt[3]{x}+\sqrt[3]{y}=3\\ x+\quad y=9\end{array}\right\}$$

and form the equation whose roots are $\frac{2}{3}$ and $-\frac{3}{2}$.

4. A carpenter agreed to work for 60 days on condition that he should receive for each day that he worked Re. 1 4 as. and his board, and pay 4 as. 8 pies for his board each day he was idle. At the end of the term he received Rs 49 9 as. 4 p. How many days did he work?

5. A and B sell a number of Cashmere shawls for Rs 284. B sells four shawls more than A, and if he had sold the quantity A sold he would have recieved Rs 100 for them, while

A would have received Rs 168 for what B sold. Find how many shawls each sold and the price per shawl which each received.

[VII]

1. State and prove the rule for finding the H. C. D. of two algebraical expressions.

Find the H. C. D of $6x^3+x^2-1$ and $4x^4-6x^3-4x^2+3x$.

2. At what time between twelve o'clock and one o'clock are the hour and minute hands of a clock exactly opposite to one another?

3. Shew that when $n+2$ figures of a cube root have been obtained by the ordinary method, n more may be obtained by division only supposing $2n+2$ to be the whole number.

4. Show how to solve the equation $ax^2+bx+c=0$, and find the necessary condition in order that its roots may be equal.

5. Solve the following equations :—

(1) $\sqrt{(1-x+x^2)}-\sqrt{(1+x+x^2)}=m$.

(2) $2^{x+1}+4^x=288$.

6. Shew that if $\dfrac{ay-bx}{c}=\dfrac{cx-az}{b}=\dfrac{bz-cy}{a}$, then $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

7. Define *proportion.*

Show that if A varies as B when C is invariable and as C when B is invariable; then will A vary as BC when both B and C are variable.

8. Define Arithmetical, Geometrical and Harmonical Progressions.

Find the sum of the cubes of the first n natural numbers.

9. Without assuming the formula for permutations, find the number of combinations of n things taken r at a time.

For what value of r will this be greatest?

10. Prove the Binomial Theorem for any exponent assuming its truth where n is a positive integer.

What is the r^{th} term of $\dfrac{1}{\sqrt{(1-x)}}$.

[VIII.]

1. Given log ·3010300 and log $3=·477121$ find the logarithms of ·05 and 5·4.

2. Resolve $a^2-b^2-c^2+d^2-2(ad-bc)$ into factors, and divide $(ay-bx)^2-(ax+by)^2$ by $(a-b)y+(a-b)x$.

3. A starts from Bombay for Poona, and B at the same time from Poona for Bombay. Each travels at a uniform rate. A reaches Poona 16 hours and B reaches Bombay 36 hours after passing each other on the road. Find in what time each has performed his journey.

4. Solve the equations

$$\left.\begin{aligned} x^2y+xy^2&=30 \\ \frac{1}{x}+\frac{1}{y}&=\frac{5}{6}. \end{aligned}\right\}$$

5. Four numbers are in A. P. Their common difference is 3, and their product 280. What are the numbers?

6. Write down the middle term of the binomial expansion $(x+y)^{12}$ and the coefficient of x^2 in the expansion $\left(x^2+\frac{a}{x}\right)^7$.

[IX]

1. Obtain the numerical value of

$$a^3+b^3+c^3-3abc,$$

when $a=\frac{1}{2}$, $b=\frac{1}{4}$, $c=\frac{3}{4}$.

2. Reduce to its lowest dimensions the fraction :—

$$\frac{3x^7-7x^3+4}{3x^7-7x^4+3}.$$

3. Solve the equations

$$(\alpha)\quad \frac{x+3}{3}-\frac{x+4}{4}+\frac{1}{x-12}=\frac{x+1}{12}$$

$$(\beta)\quad \frac{b^2-c^2}{x-a}+\frac{c^2-a^2}{x-b}+\frac{a^2-b^2}{x-c}=0.$$

$$(\gamma)\quad (x-2)\sqrt{(x^2-2x+9)}=(x+2)(3-x).$$

4. The value of two sets of diamonds is such, that if a thousand rupees be added to the price of the first, it will be

double that of the second; but if a thousand rupees be added to the price of the second, it will be two-thirds that of the first. Find the price of each set.

5. A stands in a line with 12 equidistant balls, and B leaving A, picks up the balls in succession, returning with each to A, and when he has given him the last, finds that he has walked exactly three-fourths of a mile. Had A and the balls all been equidistant, and the last ball been just as far from A as before, B would have had to walk 176 yards less. Find the distance of the balls from one another.

6. Given
$$\begin{aligned} y+z+n&=ax \\ z+n+x&=by \\ n+x+y&=cz \\ x+y+z&=dn \end{aligned}$$
prove that $\dfrac{1}{1+a}+\dfrac{1}{1+b}+\dfrac{1}{1+c}+\dfrac{1}{1+d}=1.$

7. If a, b, c, d are in continued proportion, then
$$a : d :: a^3 : b^3.$$
If $am : bf :: cn : dg :: eo : fm$, prove that
$$amdg : bcnf :: eob : am^2$$

8. In a geometrical progression show that
$$s=\frac{a(r^n-1)}{r-1}$$
find the value of the recurring decimal $\cdot\dot{4}2857\dot{1}$.

9. If a, b, c are in G. P., and $a+x$, $b+x$, $c+x$, in H. P, find the value of x.

10. Given that the number of permutations of n things taken r at a time is $n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)$ show that the number of combinations of n things taken r at a time is $\dfrac{\lfloor n}{\lfloor r\,\lfloor n-r}$. Hence deduce the value of $\lfloor 0$.

11. Apply the Binomial Theorem to find the cube root of 9 to 5 places of decimals.

12. Prove that
$$\log 2=7\log\tfrac{16}{15}+5\log\tfrac{25}{24}+3\log\tfrac{81}{80}.$$

[X]

1. What relation must obtain between the coefficients in order that a quadratic equation may have equal roots? For what value of c will the equation $4x^2+8x+c=0$ have equal roots?

2. Two steamers start at the same time one from Bombay and the other from Aden and sail at uniform rates of speed so as to meet. When they meet it is found that the Bombay vessel has sailed 330 miles further than the other. She reaches Aden 4 days after the time of meeting while the other reaches Bombay in 9 hours. Find the distance between Aden and Bombay supposing both vessels to have pursued the same course.

3. The 1st, 3rd and 9th terms of an A. P. form a G. P. the sum of which is 39. Obtain the sum of the nine terms.

4. Of four numbers the first three are in A. P. and the last three in H. P. Prove that four numbers are proportionals.

5. Find the middle term of the expansion of $(1+x)^{2n}$ and the greatest term of the expansion of $(1+\frac{5}{8})^{\frac{3}{2}}$.

6. What is the distinction between the Napierian and the common system of logarithms? Why are the Napierian called natural logarithms?

Given $\log_{10}6=\cdot7781513$ and $\log_{10}8=\cdot9030900$; obtain the logarithms of 36, 64, 75 and 125.

7. Show how logarithms may be advantageously employed in solving questions in compound interest.

Given $\log(1+x)=x-x^2+x^3-$&c. and $\log_e 2=\cdot6931472$, show that any sum of money at compound interest per annum will double itself in approximately $\frac{69}{m}$ years where m is the rate of interest per cent. per annum.

[XI]

1. If $S=a+b+c$, prove that—

$$(as+bc)(bs+ac)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2.$$

2. Distinguish between a quadratic equation and a quadratic expression, and prove that whatever real value x may have,

ax^2+bx+c, and a never differ in sign, except when the roots of $ax^2+bx+c=0$ are possible and different, and when x is taken so as to lie between them.

3. Solve the equations :—

(1) $(x+1)(x+2)(x+3)=6$

(2) $(x+3)(x-2)+\sqrt{\{(x+4)(x-3)\}}=162.$

(3) $\left.\begin{array}{l} x^3+y^3+z^3=3xyz. \\ x-a=y-b=z-c \end{array}\right\}$

4. When Compound Interest is reckoned at 5 per cent. per annum, the difference between the discount and interest on any sum for two years is $(\frac{41}{420})^2$ of the sum ?

5. If x varies jointly as y and z ; and y varies directly as $x+z$, and if $x=2$ when $z=2$, find the value of z when $x=9$.

6. If in the equation $s=\{2a+(n-1)d\}\frac{n}{2}$, n has a negative value, prove that this value corresponds to a series of $-n$ terms, having the common difference d and $d-a$ for its first term.

7. If p be the continued product of a series of terms in G. P., show that $p^2=(al)^n$ where a and l are the first and last terms.

8. Find for what value of r the number of combinations of n things taken r at a time is greatest ?

9. Find the sum of the coefficient, of the first $r+1$ terms in the expansion of $(1-x)^{-n}$.

10. Investigate the rule for obtaining the characteristic of the logarithm of any number greater or less than unity.

11. Given log $2=\cdot3010300$, log $3=\cdot4771213$, find the logarithms of the following numbers $7\frac{1}{2}$, $1\frac{4}{5}$, $\cdot0036$, $\cdot0192$, $4\cdot8$.

[XII]

1. Find the value of—

$$\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}, \text{ when } x=\frac{2ab}{b^2+1}.$$

2. If $x+y+z=0$, prove that—

$$(x^3+y^3+z^3)^3=27\,x^3\,y^3\,z^3.$$

3. Solve the following equations :—

(1) $(x-3)(x-4)(x-5)(x-6)=24.$

(2) $x^{\frac{1}{3}}+x^{-\frac{1}{3}}=\frac{4}{13}\left(x+x^{-1}\right)$

4. Given that $s \propto t^2$ when f is constant, and $s \propto f$, when t is constant : also that $2s=f$ when $t=1$. Find the equation between f, s, and t.

5. In an arithmetical progression prove that—

$$b=\frac{(l+a)\,(l-a)}{2s-(l+a)},$$

where a is the first term, b the common difference, l the last term, and s the sum of the series.

6. In a geometrical series if $l_1 . l_2$, l_3 be the n^{th}, $2n^{\text{th}}$, $3n^{\text{th}}$ terms respectively, prove that—

$$l_2{}^2=l_1\,l_3.$$

7. If a, b, c be three quantities such that a is the arithmetic mean between b and c, and c the harmonical mean between a and b, show that b is the geometrical mean between a and c, and compare a, b, c.

8. Find for what value of r the number of combinations of n things taken r at a time is greatest.

9. Prove that if n be any positive integer, the integral part of $(2+\sqrt{3})^n$ is an odd number.

10. Find the $(r+1)^{\text{th}}$ term in the expansion of $(1-x)^{-3}$.

11. Write down the 8th term of $(1-x)^{-\frac{1}{6}}$ and find the greatest term in the expansion of $(1+\frac{1}{3})^{18}$.

[XIII]

1. Find the value of—

$$\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$$

$$\text{when } x=\frac{12ab}{a+b+\sqrt{\{(a+b)^2+12ab\}}}.$$

2. Solve the equation—

$$x^4+\tfrac{1}{4}=x\sqrt{(2)}\,\sqrt{(x^4-\tfrac{1}{4})}.$$

3. If α, β be the roots of the equation $x^2 - px + q = o$, prove that $\alpha + \beta = p$ and $\alpha\beta = q$ and that the roots of the equation $x^2 - (p^2 - 2q)x + q^2 = 0$ are the squares of the roots of the original equation.

4. The attendance at Professor's lectures varies directly as the Professor's power of exposition and inversely as the square of the number of lectures delivered. If sixty-four students attend the lectures of Professor A who delivers a course of twelve lectures, find the number of students who attend the lectures of Professor B who delivers a course of sixteen lectures and who possesses twice as much power of exposition as Professor A.

5. If $a : b :: c : d :: e : f$, prove that—

$$\frac{a^2 + c^2 + e^2}{ab + cd + ef} = \frac{ab + cd + ef}{b^2 + d^2 + f^2}.$$

6. Find an arithmetical progression such that the sum of n terms shall be equal to n^2.

7. In a geometrical progression prove that—

$$r^n - \frac{s}{s - l} r^{n-1} + \frac{l}{s - l} = 0.$$

8. Prove that the number of combinations of n things taken r at a time is the same as the number of them taken $n - r$ at a time.

9. Find the $(r+1)^{th}$ term in the expansion of $(1 + x)^{-2}$.

10. Find the two middle terms of $(a + x)^{13}$ and show that the middle term in the expansion of $(1 + x)^{2n}$ is—

$$\frac{1 \cdot 3 \cdot 5 \ldots\ldots\ldots (2n-1)}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots n} 2^n x^n.$$

[XIV]

1. If $2p = a + b + c$ and $2q^3 = a^3 + b^3 + c^3$, show that $(q^3 - a^3)(p - a) + (q^3 - b^3)(p - b) + (q^3 - c^3)(p - c) = a^4 + b^4 + c^4 - q^3 p$.

2. Show how to solve a quadratic equation in its most general form.

A gentleman bought a horse for a certain sum, and having re-sold it for £119 found that he had gained as much per cent. by the transaction as the horse cost him. What was the prime cost of the horse?

3. Distinguish between *arithmetical* and *geometrical* progression: and prove that $s=\frac{ar^n-a}{r-1}$, where s=sum of the series, a=first term, and r=common ratio; and state to what series the formula belongs.

4. If a steam engine is observed to pass over 4 feet in the first second, and 88 feet in the 60th second of its motion, how far will it travel in the first minute?

5. Sum to infinity the series $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\frac{1}{4\cdot5}+$&c.

6. Define the term *combination*, and prove the formula for the number of combinations of n things taken r at a time.

7. Find the sum of the coefficients of the terms in the expansion of $(1+x)^n$; and show that the sum of the coefficients of *odd* terms is equal to the sum of the coefficients of *even* terms$=2^{n-1}$.

8. Write down the fourth term of $(3a-2x)^{\frac{1}{2}}$ and the eighth term of $(1-x)^{-\frac{1}{6}}$.

[XV].

1. If $A=qB+C$, show that the highest common divisor of B and C is also that of A and B.

Find the highest common divisor of—

$$16x^4-53x^2+45x+6 \text{ and } 8x^4-30x^3+31x^2-12.$$

2. Simplify $\frac{15}{\sqrt{(10)}+\sqrt{(20)}+\sqrt{(40)}-\sqrt{(5)}-\sqrt{(80)}}$.

3. If α and β be the roots of the equation $ax^2+bx+c=0$, form the equation whose roots are $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$, and find the value of $\alpha^3+\beta^3$.

4. If $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=$&c., show that each is equal to $\frac{a_1+a_2+a_3+\dots}{b_1+b_2+b_3+\dots}$ and also to $\frac{(a_1p+a_2p+a_3p+\dots)^{\frac{1}{p}}}{(b_1p+b_2p+b_3p+\dots)^{\frac{1}{p}}}$.

If $\frac{a+bx}{b+cy}=\frac{b+cx}{c+ay}=\frac{c+ax}{a+by}$, then each $=\frac{1+x}{1+y}$.

5. Show how to find the sum of a given number of quantities in geometrical progression, the first term and the common ratio being supposed known.

Hence deduce the usual rule for finding the value of a recurring decimal.

6. The p^{th} term of an arithmetical progression is q, and the q^{th} term is p; find the $(p+q)^{th}$ term.

7. Find the number of homogeneous products of r dimensions that can be formed out of n letters abc......... and their powers.

[XVI]

1. Define an *expression.* When is an expression said to be *homogeneous* ?

Shew that the value of the expression :—

$$y^2z - z^2y + z^2x - x^2z + x^2y - xy^2$$

is not altered if any the same quantity be added to or subtracted from each of the quantities x, y and z.

2. Prove that the square root of a binomial, one of whose terms is a quadratic surd and the other rational, may sometimes be expressed by a binomial, one or each of whose terms is a quadratic surd. In what case is it useless to employ this method ?

Find the square root of $a+b+\sqrt{(2ab+b^2)}$.

3. Find the sum and product of the roots of the quadratic—

$$px^2+qx+r=0.$$

If α and β be the roots of this equation, shew that the roots of the equation—

$$qrx^2+(pr+q^2)x+pq=0,$$

$$\text{are } \frac{1}{\alpha+\beta} \text{ and } \frac{1}{\alpha}+\frac{1}{\beta}.$$

4. Prove that a ratio of greater inequality is diminished and of less inequality increased by adding any quantity to both its terms.

If four numbers be proportionals, shew that there is no number which being added to each will leave the resulting four numbers proportionals.

5. Find the sum of a G. P. to n terms and when possible to infinity.

If s_1, s_2, s_3 be the sums to n, $2n$, $3n$ terms respectively, prove that—

$$s_1(s_3 - s_2) = (s_2 - s_1)^2.$$

6. If $(n)_r$ represent the number of combinations of n things taken r together, prove independently of any formula that—

$$r(n)_r = n(n-1)r-1.$$

Four persons are chosen by lot out of ten; in how many ways can this be done and how often would any one person be chosen?

7. If m be any quantity whatever and $f(m)$ represent the series—

$$1 + mx + \frac{m(m-1)}{\lfloor 2}x^2 + \frac{m(m-1)(m-2)}{\lfloor 3}x^3 + \ldots\ldots$$

prove that $f(m) \times f(n) = f(m+n)$. By what consideration does Euler prove this relation and on what principle does he base his demonstration? By the aid of this formula prove the Binomial Theorem for a *positive fractional* exponent.

[XVII]

1. Given $\log 2 = \cdot 30103$ and $\log 3 = \cdot 47712$, find the logarithm of 12 to the base 40.

2. Given $\log x - \log \sqrt{x} = \dfrac{2}{\log x}$; find x.

3. Simplify the expressions

(a) $$\frac{1}{x-1} + \frac{2}{2x+1-\sqrt{(-3)}} + \frac{2}{2x+1+\sqrt{(-3)}}$$

(β) $$\frac{\dfrac{1}{a} + \dfrac{1}{b+c}}{\dfrac{1}{a} - \dfrac{1}{b+c}}\left\{1 + \frac{b^2+c^2-a^2}{2bc}\right\}.$$

4. Find the values of x and y from the equations

(a) $$\left.\begin{aligned} x^2 + 3y + x &= 75 - 2xy \\ y^2 - y + x &= 24 \end{aligned}\right\}$$

(β) $$\left.\begin{aligned} x^4 &= 3x + 2y \\ y^4 &= 2x + 3y \end{aligned}\right\}$$

5. A and B engage in speculation. A disposes of his shares for Rs110 and his gain per cent. is equal to $\frac{1}{10}$ of B's investment, B's gain is Rs360 and it appears that A gains four times as much per cent. as B. Find the original capital invested by each.

6. Sum the following series to n terms

$$1 \cdot 2 \cdot 4 + 2 \cdot 3 \cdot 5 + 3 \cdot 4 \cdot 6 + \text{\&c. to } n \text{ terms}$$

$$\text{and} \quad \frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 6} + \text{\&c. to } n \text{ ferms.}$$

7. A man owes a certain number of Rupees which he pays in one year by weekly instalments. He pays Re. 1 the first week, Rs2 the second week, Rs5 the third week, and so on. Find the amount of the debt and the last payment.

8. Find the number of permutations that can be formed out of the letters of the words—*Matriculation*, *University*.

[XVIII]

1. Express $\sqrt{\{\sqrt[3]{(a+b)}\} \times \sqrt[3]{\{\sqrt{(a+b)}\}}}$ as a single surd.

2. Solve the equations :—

$$\left.\begin{aligned} \sqrt{\left(\frac{x}{y}\right)} + \sqrt{\frac{y}{x}} = \frac{61}{\sqrt{(xy)}} + 1 \\ (x^3y)^{\frac{1}{4}} + (xy^3)^{\frac{1}{4}} = 78. \end{aligned}\right\}$$

3. The area of a square field exceeds that of a circular field by one acre, and the boundary of the former is 400 yards longer than that of the latter; find their dimensions.

4. With 8 consonants and 4 vowels, how many words can be spelled, each word to contain 3 consonants and 2 vowels?

5. Find the thirteenth term of $\left(x - \frac{2y^2}{2x}\right)^{15}$.

6. What relation must exist between a, β, γ, in order that they shall be the p^{th}, q^{th} and r^{th} terms of an A.P.

7. If a, b, c, d be any consecutive coefficients of an expanded binomial; show that $(bc+ad)(b-c) = 2(ac^2 - b^2d)$.

8. A wine merchant bought a cask of wine for Rs480 and sold a quantity exceeding by two gallons three fourths of the whole at a profit of 25 per cent. He subsequently sold the

remainder at such a price that he cleared 60 per cent. on the whole transaction; but if he had sold the whole quantity at the latter price, he would have gained 175 per cent. How many gallons were in the cask?

[XIX]

1. Solve the equations :—

$$\left.\begin{aligned} x^2+yz+xz&=29 \\ y^2+xz+xy&=25 \\ z^2+xy+yz&=46 \end{aligned}\right\} \qquad \text{(i)}$$

$$\left.\begin{aligned} x^2+\ yz&=\ 6y-11 \\ y^2+\ xz&=\ 6x-\ 2 \\ z^2+8xy&=12z+16 \end{aligned}\right\} \qquad \text{(ii)}$$

2. Sum to n terms the series

$$2+4a+6a^2+10a^3+18a^4+34a^5+66a^6+\ldots\ldots$$

3. The coefficients of x in the third and fifth terms of $(1-x)^n$ are $\frac{14}{9}$ and $-\frac{7}{243}$ respectively; find n.

4. There are three numbers in A. P. the sum of which $=21$, and to which if 2, 5 and 14 be added, respectively, the sums are in G. P., find the numbers.

5. Out of 7 Parsees and 8 Hindoos 5 are to be selected which must always consist of 3 Parsees and 2 Hindoos; in how many ways may the selection be made.

6. A caught B asleep with a cask of wine beside him; seizing the opportunity he drank the wine for two-thirds of the time that B would take to empty the cask; at this moment B awoke and finished what remained. If both had commenced to drink together, the wine would have been consumed two hours earlier, but in this case A would only have got half of what he left for B. In what time would each finish the cask alone, and in what time together?

[XX]

1. If α and β be the roots of the equation $Ax^2+Bx+C=0$, prove that $\alpha+\beta=-\dfrac{B}{A}$ and $\alpha\beta=\dfrac{C}{A}$, and thence find the value of $\alpha^4+\alpha^2\beta^2+\beta^4$.

2. Show that the solutions of the equation $Cx^2+Bx+A=0$ are the reciprocals of those of the equation $Ax^2+Bx+C=0$.

3. Out of 17 consonants and 5 vowels all different, how many words can be formed each consisting of two vowels and three consonants, supposing that a vowel is placed between two consonants?

4. A and B engaged to perform equal quantities of work. A commenced work half an hour before B, and both rested from 12 to 1 o'clock, having completed just half the whole amount of work. B completed his task at 7 P. M., but A was obliged to remain until $10\frac{1}{4}$ P. M. in order to complete his. At what hour in the morning did A commence work?

5. In the two series 2, 5, 8, &c. and 3, 7, 11 &c. each continued to 100 terms, find how many terms are common to both series.

6. If from a vessel containing a gallons of wine y gallons be drawn off and the vessel filled with water, and this be repeated n times, what quantity of wine will remain in the vessel after the n^{th} operation?

[XXI]

1. If the equation $x^2-2\dfrac{m+n}{1+mn}x+1=0$ have real roots,

then the equation $(x+m)(x+n)+k\left(x+\dfrac{1}{m}\right)\left(x+\dfrac{1}{n}\right)=0$

will have real roots for all values of k.

2. Determine in what cases the rule for finding the greatest term in the expansion of $(1+x)^n$ when n is a positive integer will also apply when n is positive but not integral.

3. Investigate the sum of n terms of the series whose n^{th} term is $n(n+1)(n+2)......(n+r-1)$.

Sum the series

$$\frac{6}{2.3.4}+\frac{10}{3.4.5}+\frac{14}{4.5.6}+\&\text{c. to } n \text{ terms and to infinity.}$$

4. Find the sum of n terms of the series of cube numbers $1^3+2^3+3^3+4^3+\&c.$

5. There is a fraction such that if its numerator be increased and its denominator diminished by 2, the reciprocal of the fraction will be the result; while, if the numerator be diminished and the denominator increased by 2, a result will be obtained less than the reciprocal by $1\frac{1}{15}$. What is the fraction?

6. The forewheel of a carriage makes 6 revolutions more than the hind-wheel in 120 yards; but the former would only make 4 revolutions more than the latter if the spokes of each wheel lengthened 5 inches. Find the circumference of each wheel.

[XXII]

1. Prove that $(n+1)(n+2)(n+3)$.........to n factors $=2^n \times 1.3.5$.........to n factors.

2. Show that a factor may be found which will rationalize any binomial.

Reduce $\dfrac{\sqrt{(1+x)}-\sqrt{(1-x)}}{\sqrt{(1+x)}\sqrt{(1-x)}}$ where $x=\dfrac{\sqrt{3}}{2}$.

3. Distinguish between a quadratic *equation* and a quadratic *expression* and shew that a quadratic equation has two and only two roots.

Solve the equation $\sqrt{(x^2-3x+8)}=\frac{3}{4}(x^2-7x+8)$.

4. Define *ratio, variation, proportional, commensurable.*

If A vary as B when C is constant and A vary as C when B is constant, prove that A will vary as the product BC when both B and C are variable. Give fully a geometrical illustration.

5. Sum the series :—

(1) $17\frac{1}{2}$, $14\frac{1}{6}$, $10\frac{5}{6}$..................to 24 terms.

(2) 1, 3, 6, 10, 15, ...to n terms.

(3) $\frac{5}{7}$, $\frac{7}{21}$, $\frac{9}{63}$, $\frac{11}{189}$, ...to infinity.

6. Investigate a formula for the number of permutations of n things taken all together which are not all different.

How many different numbers can be made out of all the figures of 111223?

7. Enunciate the *Binomial Theorem* and prove it for a positive integral exponent.

Write down the $r+1^{\text{th}}$ term in the expansion of $\sqrt{(a^2-x^2)}$ and the coefficient of x^{10} in $\dfrac{1+x}{(1-x)^3}$.

[XXIII]

1. If $3s=a+b+c$, prove that

$$(s-a)^4+(s-b)^4+(s-c)^4$$
$$=2\{(s-a)^2(s-b)^2+(s-b)^2(s-c)^2+(s-c)^2(s-a)^2\}.$$

2. Shew that if two quadratic surds cannot be reduced to others which have the same irrational part, their product is irrational, and also that one quadratic surd cannot be made up of two others which have not the same irrational part.

3. If $a : b = c : d$, shew that

(α) $ma + nb : pa + qb = mc + nd : pc + qd$.

(β) $\frac{1}{ma} + \frac{1}{nb} + \frac{1}{pc} + \frac{1}{pd} = \frac{1}{bc}\left(\frac{a}{q} + \frac{b}{p} + \frac{c}{n} + \frac{d}{m}\right)$.

4. What do you understand by the *limit of an infinite series in Geometrical Progression*? Are Arithmetical series susceptible of limits? Find the sum of an infinite G. P.

Determine the value of $\sqrt{a^3\sqrt[3]{b\sqrt{a^2\sqrt[3]{b \ldots\ldots\ldots}}}}$

continued to infinity.

5. Insert a given number of Harmonical means between two given terms.

If a, b, c, be in G. P. and $a^x = b^y = c^z$, prove that x, y, z, are in H. P.

6. Find for what value of r the number of combinations of n things taken r at a time is greatest.

A Bráhman, hospitably disposed, wishes to make up as many different parties as he can out of 40 friends, each party consisting of the same number; how many should he invite at a time?

7. Investigate the sum of the coefficients of the terms in the expansion of $(1+x)^n$ and shew that the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms, n being a positive integer.

Prove that if n be a positive integer.

$$(1+x)^n(1+x^n) > 2^{n+1}x^n.$$

Miscellaneous Examination Papers.

A.

1. Multiply $a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+a^{\frac{1}{2}}b+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$

$$\text{by } a-2a^{\frac{1}{2}}b^{\frac{1}{3}}+b^{\frac{2}{3}}.$$

Multiply $a^{\frac{5}{6}}-a^{\frac{5}{12}}b^{\frac{5}{8}}+b^{\frac{5}{4}}$ by $a^{\frac{5}{6}}+a^{\frac{5}{12}}b^{\frac{5}{8}}+b^{\frac{5}{4}}$ and prove that the result is a factor of

$$a^{\frac{10}{3}}-a^{\frac{5}{3}}b^{\frac{5}{2}}+b^5.$$

2. Find the G. C. M. of—

$$2y^3+y^2-8y+5, \text{ and } 7y^2-12y+5.$$

3. Reduce the following expressions to their simplest forms :—

(a) $\dfrac{x^4+x^2y^2+y^4}{x^4+2x^3y+3x^2y^2}+2xy^3+y^4.$

(b) $\dfrac{1}{(a-b)(a-c)(x+a)}+\dfrac{1}{(b-a)(b-c)(x+b)}$

$$+\frac{1}{(c-a)(c-b)(x+c)}.$$

4. Solve the following equations :—

(a) $\dfrac{10x-7}{18}-\dfrac{12x+2}{11x-8}=\dfrac{5x-4}{9}.$

(b) $\left.\begin{aligned} x-y&=2 \\ \frac{x}{y}-\frac{y}{x}&=1\tfrac{1}{15} \end{aligned}\right\}$

(c) $\dfrac{x-1}{x+1}+\dfrac{x+3}{x-3}=2\left(\dfrac{x+2}{x-2}\right).$

(d) $a^x-8a^{-x}=2.$

5. Having bought as many muskets as cost £8,500, I reserved 800, and sold the remainder for £8,400, gaining 10*s.* on each. How many did I purchase, and what did each cost ?

6. A farmer sells to one person 9 horses and 7 cows for £300; and to another 6 horses at an advance of $12\frac{1}{2}$ per cent. on

the former price, and 13 cows at an advance of 8 per cent. for £$330\frac{13}{25}$. What was the price of each?

7. Sum the series—

(a) $\sqrt{(2)} - 2 + 2\sqrt{(2)} - \&c.$, to n terms.

(b) $1 + 4p + 7p^2 + 10p^3 + \&c.$, to n terms.

8. Find the co-efficient of x^n in the expansion of—

(a) $\sqrt{\{(a^5 - b^3x^2)^5\}}$.

(b) $\sqrt[3]{\left\{\dfrac{a^2}{(a-x)^4}\right\}}$.

B

1. State the method for finding the G. C. M. of two Algebraical expressions. What artifices are permissible or employed to shorten the process?

Find the G. C. M. of

$$16x^4 - 8x^3 + 8x^2 + 2x + 3,$$
$$16x^4 + 16x^3 - 2x - 2$$

and the L. C. M. of

$$x^3 + a^3,\ x^5 + a^5,\ x^2 - ax + a^2,\ x^2 + ax + a^2.$$

2. Simplify the following expression—

$$\frac{a+b}{a-b} + \frac{b+c}{b-c} + \frac{c+a}{c-a} + \frac{(a+b)(b+c)(c+a)}{(a-b)(b-c)(c-a)}.$$

3. Prove that

$$\left(x+\frac{1}{x}\right)^2 + \left(y+\frac{1}{y}\right)^2 + \left(z+\frac{1}{z}\right)^2$$
$$= 4 + \left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right),$$

if $xyz = 1$.

4. Extract the square root of

$$5x^6 - 4\sqrt{(5)}\,x^5 + 2\{(2+\sqrt{(15)}\}x^4 - 2\{(2\sqrt{(3)} + \sqrt{5})\}x^3$$
$$+ 7x^2 - 2\sqrt{(3)}x + 1.$$

5. Solve the following equations :—

(a) $\sqrt{(x^2 + x + 1)} + \sqrt{(x^2 - x + 1)} = 2$

(b) $\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}=b$

(c) $x^2-x+\sqrt{(x^2-7x+8)}=6x+4$

(d) $$\begin{aligned}5x-2y+z&=20\\ x+2y+5z&=28\\ 4x+5y-7z&=19.\end{aligned}$$

6. Sum the following series :—

(a) $\frac{5}{6}+\frac{2}{3}+\frac{7}{6}+$ &c., to 40 terms.
(b) $1+11+101+1001+$ &c., to n terms.
(c) $(1-n)-(1-n)^2+(1-n)^3-$ *ad infin.*

7. Expand $(2x-3a^2)^7$; $(x^{\frac{1}{3}}-2y^{-\frac{5}{6}})$ to five terms : and find the 12th term of $(x^2-5ay^3)^{15}$.

8. In how many ways can a pack of cards be arranged so that all the cards of one kind may—1st, be consecutive ; and 2ndly, consecutive, and in proper order.

C

1. Define "Greatest Common Measure" and "Least Common Multiple." Find the G. C. M. of—

$$12x^2-15yx+3y^2 \text{ and } 6x^3-6yx^2+2y^2x-2y^3$$

2. Find the values of x in the following equations

$$\frac{x^3+1}{x^2-1}=x+\sqrt{\left(\frac{6}{x}\right)}\ ;\ x^4-2x^3+x=132.$$

3. If the numerator of a fraction is increased by one, its value becomes one-third ; if the denominator is increased by one its value becomes one-fourth. What is the fraction ?

What are eggs a dozen when two more in eight annas worth lowers the price eight pies per dozen ?

4. Prove that the sum of any number of terms in Arithmetical Progression is equal to the product of the number of terms into half the sum of the first and last terms.

How many terms of $16+24+32+40$, &c., amount to 1840 ?

5. The Arithmetical mean of two numbers is 3, and the Harmonical mean is $\frac{8}{3}$. Find the numbers.

There are four numbers, the three first of which are in G. P. and the last three in A. P.; the sum of the first and last is 14, and that of the second and third is 12. Find the numbers.

6. Define "Variations," "Permutations," "Combinations." In how many ways can the crew of an eight-oared boat be arranged, when three men can pull only on one side, and two only on the other.

If each man can pull on either side, in how many ways can they then be arranged?

D

1. Explain the use of brackets in algebra and if $a=2$, $b=3, c=6, d=5$. Find the value of—

$$a+2c-\{b+d-[a-c-(b-2d)]\}.$$

2. Simplify—

$$\frac{3}{1-2x}-\frac{7}{1+2x}-\frac{4-20x}{4x^2-1}.$$

3. A crew rowing a boat on the canal find they can row down-stream three times as fast as up-stream. They row from Roorkee to Dhunowrie (6 miles) and back again in two hours. What would be the pace of the boat in still water?

4. Solve the equation—

$$\{(7-4\sqrt{3})\}x^2+\{(2-\sqrt{3})\}x=2$$

and form the quadratic equation whose roots are 4 and 5.

5. In the equation $a^n=b$, which is the logarithm, which the base? Prove that the logarithm of a product is equal to the sum of the logarithms of its factors.

6. Given $\log 2=\cdot3010300.$

$\log 3=\cdot4771213.$

find the logs of 36, 27 & 16.

E.

1. Find the value of $\frac{1}{\sqrt{5}}$ to six places of decimals.

2. Prove that the logarithm of a product is the sum of the logarithms of the factors of the product.

Given $\log 2=\cdot3010300$, and $\log 6=\cdot7781513$, find $\log 15$ and $\log \cdot0025$.

3. Multiply together $x^2-(b-c)x-bc$, $x^2-(c-a)x-ca$, $x^2-(a-b)x-ab$; and divide the result by $x^3-(a+b+c)x^2+(bc+ca+ab)x-abc$.

4. Reduce to their simplest forms

(i) $$\frac{x+1}{x^2+5x+6}-\frac{2(x+2)}{x^2+4x+3}+\frac{x+3}{x^2+3x+2};$$

(ii) $\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}\times\{a^2(b+c)+b^2(c+a)+c^2(a+b)\}-\{a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)\}$.

5. Find the sum of an odd number of terms of an arithmetical progression, given the middle term and the number of terms.

Sum n terms of each of the series

(i.) $a+b+(a-b)$, $a+b+2(a-b)$, $a+b+3(a-b)$, $a+b+4(a-b)$.........

(ii.) $a+b$, $a-b$, $\dfrac{(a-b)^2}{a+b}$, $\dfrac{(a-b)^3}{(a+b)^2}$..................

6. Solve the equations

(i.) $\dfrac{x-1}{x-2}-\dfrac{2}{x-1}=1$; (ii.) $x+\dfrac{1}{x}=2\dfrac{1}{1640}$.

7. To do a piece of work A requires $\frac{5}{6}$ as long as B; but A's wages are 1 shilling a day more than B's. The work costs 22*s.* more if A does it than if B does it, and if they work together it costs 10 guineas. Find the daily wages of each.

F

1. Find to five decimal places the value of $\dfrac{\sqrt{(7)}-\sqrt{(5)}}{\sqrt{(7)}+\sqrt{(5)}}$.

2. Reduce to its simplest form the following expression:

$$\frac{x^2+3x+2}{x^2+4x+3}+\frac{x^2-3x+2}{x^2-4x+3}+\frac{x^2-12x+35}{x^2-2x-15}.$$

3. Being given ten consonants and five vowels, find the number of words that could be formed out of them, each word containing three of the consonants and two of the vowels.

4. In boring a well, 400 feet deep, the cost is 2*s.* 3*d.* for the first foot, and an additional penny for each foot following. What is the cost of boring the last foot, and also of boring the entire well?

5. Express $u^3+v^3+w^3-3uvw$ in terms of a, b, c; being given $u=b+c-a$, $v=c+a-b$, $w=a+b-c$.

6. Solve the equations—

(i) $\frac{2x-1}{3x+2}+\frac{2x-3}{2x-1}+\frac{2x-1}{6x^2+x-2}=0$;

(ii) $x^4+x^3-4x^2+x+1=0$.

7. Find three numbers in geometrical progression, their product being 1728, and the sum of the extremes 51.

8. In what time would a sum of money accumulating at 3 per cent. per annum compound interest come to five times its original amount?

[Given $\log 2=\cdot30103$, $\log 2575=3\cdot41078$.]

9. Find two numbers such that their sum is 9 and the sum of their fourth powers 2417.

G.

1. Simplify the following expressions :—

$$\frac{a^3}{(a-b)(a-c)(x-a)}+\frac{b^3}{(b-c)(b-a)(x-b)}+\frac{c^3}{(c-a)(c-b)(x-c)}.$$

2. Solve the following equations :—

(a). $\dfrac{25-\frac{x}{3}}{x+1}+\dfrac{16x+4\frac{1}{5}}{3x+2}=5+\dfrac{23}{x+1}$.

(b). $8\sqrt{(3x)}+\dfrac{243+324\sqrt{(3x)}}{16x-3}=16x+3$.

(c). $2x\sqrt{(1-x^4)}=1+x^4$.

(d). $x+y=(x-y)^2$
$x^2+y^2=\frac{5}{4}(x^2-y^2)$.

3. A and B start to run a race to a certain post and back again. A returning meets B 90 yards from the post and arrives at the starting place 3 minutes before him. If he had set out again immediately to meet B, he would have met him at a 6th of the distance between the post and the starting place. Find the length of the course and duration of the race.

4. Extract the square root of

$$(3+\sqrt{5})(x^4+1)+2(1+\sqrt{5})(x^3+x)+(8+2\sqrt{5})x^2.$$

5. Sum the following series :—

(*a*) $3+9+15+$&c. to 25 terms.

(*b*) $1.3.5+2.4.6+3.5.7+$&c. to 10 terms, and to n terms.

(*c*) $1+(1+x)+(1+x+x^2)+$&c. to n terms.

(*d*) $ac+(a+b)(a+c)+(a+2b)(a+2c)+$&c. to n terms.

6. Write down the complete expansion of

$$(1-2x)^5 \;;\text{ and } (a-bx^3)^7$$

and find the co-efficients of x^n in the expansions of

$$(2-3x)^{-4} \;;\; (1-px)^{-\frac{2}{9}} \text{ and } \frac{(1-x)^2}{(1+3x)^5}.$$

7. How many different working parties of 7 men and 2 officers can be formed out of a detachment of 25 men and 6 officers ?

If the number of combinations of $\frac{n}{3}$ things taken three together is to the number of combinations of $\frac{n}{2}$ things taken four together as 4 : 15 ; find n.

or

How many different arrangements of 3 Gentlemen and 3 Ladies (to be seated at a round table in the usual manner) can be made from a party of 7 Gentlemen and 5 Ladies?

H

1. Define "rational" and "surd" quantities and "similar surds" and when is a surd in its simplest form.

2. Multiply the following quantities :—

$$2\sqrt{(14)} \text{ by } 3\sqrt[3]{(4)}$$

and $\{3+\sqrt{(5)}\}$, $\{\sqrt{(2)}+\sqrt{(3)}\}$, $\{\sqrt{(2)}-\sqrt{(3)}\}$, $\{3-\sqrt{(5)}\}$; also divide the following quantities :—

$$16x-\frac{y^4}{16} \text{ by } 2x^{\frac{1}{4}}-\frac{y}{2} \text{ and } a^2-b \text{ by } a^{\frac{1}{3}}-b^{\frac{1}{6}}.$$

3. Define the terms "equation," "power of an equation" and "root of an equation" and solve the following

$$\frac{x-1}{\sqrt{x}+1}=4+\frac{\sqrt{x}-1}{2}$$

$$\text{and } \frac{1+x^3}{(1+x)^2}+\frac{1-x^3}{(1-x)^2}=a.$$

4. Determine the values of x which will satisfy the following expressions :—

$$\frac{x-4}{\sqrt{x}+2}=x-8$$

and $x^{\frac{7}{3}}=56x^{-\frac{2}{3}}+x^{\frac{5}{3}}$.

5. Solve the following problems :—

A person travelled 105 miles and observed that if he had travelled slower by 2 miles an hour he would have been 6 hours longer in completing the same distance. How many miles did he travel per hour?

The continued product of four consecutive numbers is 3024; find the numbers.

6. Define "arithmetical," "geometrical," and "harmonical" progression; and prove that the reciprocals of quantities in harmonical progression are in arithmetical progression.

7. Prove the rule for the sum of a geometrical series, and deduce the value in the case of infinity.

Sum the following series :—

$\frac{1}{2}+\frac{1}{3}+\frac{2}{9}$ &c..................to 7 terms.

and $2-\frac{4}{3}+\frac{8}{9}-$&c. ....to infinity.

8. Prove the rule for the value of a sum of money, at compound interest, algebraically.

If £500 be allowed to accumulate, at compound interest, at the rate of 5 per cent. per annum; what will be the amount at the end of 21 years?

I

1. Divide $ax^{-1}+ax+2$ by $a^{\frac{1}{3}}x^{-\frac{1}{3}}+a^{-\frac{1}{3}}x^{\frac{1}{3}}-1$.

2. If α and β are the roots of the equation $ax^2+bx+c=0$, prove that $\alpha+\beta=-\frac{b}{a}$, and $\alpha\beta=\frac{c}{a}$, and thence find the value of $\alpha^4+\alpha^2\beta^2+\beta^4$.

3. Solve the following equations

(a) $(x^2+1)^2=4(2x-1)$.

(b) $$\frac{4}{x^2-2x}=\frac{2}{x^2-x}+x^2-x.$$

(c) $$\frac{5}{x+2}+\frac{5}{x+4}=\frac{1}{x+1}+\frac{8}{x+3}+\frac{1}{x+5}.$$

(d) $$xyz=a(y^2+x^2)=b(x^2+z^2)=c(z^2+y^2).$$

4. If the sum of m terms of an A. P. be to the sum of n terms as $m^2 : n^2$, prove that the mth term is to the nth term as $2m-1 : 2n-1$.

Prove that $b+c$, $c+a$, $a+b$, are in H. P., if a^2, b^2, c^2, are in A. P.

5. Sum

$$1.2.4+2.3.5+3.4.6+\&c., \text{ to } n \text{ terms}$$

$$\text{and} \frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\&c., \text{ to } n \text{ terms.}$$

6. If from a vessel containing n gallons of wine, y gallons be drawn off and the vessel filled with water, and this be repeated n times; find the quantity of wine remaining after the nth operation.

7. Out of 17 consonants and 5 vowels all different, how many words can be formed, each consisting of 2 vowels and 3 consonants, supposing that a vowel was placed between two consonants.

The number of ways in which mn different things can be distributed among m persons, so that each person shall have n of them

$$\text{is } \frac{|\underline{mn}}{(|\underline{n})^m}.$$

8. Prove the Binomial Theorem for a positive index, and expand to five terms the binomials

$$\frac{1}{(a^{\frac{1}{3}}-x^{\frac{1}{3}})^6},\ \frac{3a}{(ax-x^2)^{\frac{1}{2}}},\ \left(x-3y\sqrt{(-1)}\right)^n.$$

J

1. Find the G. C. M. of—

$$y^3-(2a+b)y^2+(2ab+a^2)y-a^2b$$

and

$$3y^2-(4a+2b)\,y+(2ab+a^2)$$

and of

$$x^3+y^3+z^3 \text{ and } x^5+y^5+z^5 \text{ if } x+y+z=0.$$

2. Solve the following equations—

(a). $\sqrt{(a+bx+cx^2)}+\sqrt{(a-bx+cx^2)}=d.$

(b). $\sqrt{(1+x)}+\sqrt{\{1+\sqrt{(1+x)}\}}=11.$

(c). $x+y=5$

$$\frac{x}{y}-\frac{y}{x}=3\tfrac{3}{4}$$

(d). $\frac{y}{x}+x=\frac{5}{2}$

$$\frac{x}{y}+y=3$$

3. Simplify the following expressions—

$$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}$$

and $$\frac{2+\sqrt{(3)}}{\sqrt{(2)}+\sqrt{\{2+\sqrt{(3)}\}}}+\frac{2-\sqrt{(3)}}{\sqrt{(2)}-\sqrt{(2)}-\sqrt{(3)}}.$$

4. What number is that whose square is less by unity than the number itself?

An express train sets off to travel from one station to another with uniform speed; at the end of the first hour an accident occurs, which delays it 1 hour, and reduces its speed in the rate 5 : 3. It arrives at the second station 3 hours behind time; if the accident had occurred 50 miles further on, the train would have arrived $1\frac{1}{3}$ hour sooner. What was the distance between the stations?

5. Sum the following series—

$1+3+5$, &c., to n terms.

$2{\cdot}5+3{\cdot}6+4{\cdot}7+$&c.

6. Give the general expression for the expansion of $(1+x)^n$ employ it to find the first six terms for the expansion of—

$$(1-x)^{-2} \text{ and } (a^{\frac{2}{3}}+n^{\frac{2}{3}})^{-\frac{3}{4}}.$$

What is the co-efficient of x^{2m} in the expansion of

$\{a^2+b^2-x^2\}^{\frac{2}{3}}$ and of x in the expansion of $\dfrac{1-x}{1+x+x^2}$?

7. How may possible different signals can be made with five flags.

How many different permutations can be made with the Honor Cards, in a pack of cards, and in how many of these will the four aces be consecutive cards.

If the number of combinations of $2n$ things taken n together is to the number of combinations of $2n-2$ things taken $n-1$ together as 11 : 3; find the number of things.

K.

1. Simplify

$$\frac{(1-a^2)(1-b^2)(1-c^2)-(c+ab)(b+ac)(a+bc)}{1-a^2-b^2-c^2-2abc}$$

2. Find the square root of $10\frac{1}{2}+2\sqrt{(5)}$.

3. Find the value of

$$uv-\sqrt{(1-u^2)}\sqrt{(1-v^2)} \text{ when } 2u=x+\frac{1}{x},\ 2v=y+\frac{1}{y}.$$

4. Solve $\dfrac{1+x^3}{(1+x)^3}+\dfrac{1-x^3}{(1-x)^3}=a.$

5. Solve $\sqrt[5]{(a+x)^2}+\sqrt[5]{(a-x)^2}=3\sqrt[5]{(a^2-x^2)}$.

6. Solve $(x^2-5)^2=(x-3)^2+(x+1)^2$

7. Solve

$$\left.\begin{aligned}(x^2+y^2).\frac{y}{x}&=8\tfrac{2}{3}\\(x^2-y^2).\frac{x}{y}&=7\tfrac{1}{2}\end{aligned}\right\}$$

8. Show that there is *no* number which being added to four *unequal* numbers in proportion leaves the four results also in proportion.

9. There are five quantities (a_1 to a_5) of which a_1, a_2, a_3, are in A. P.; a_2, a_3, a_4 are in G. P.; a_3, a_4, a_5 are in H. P.; show that a_1, a_3, a_5 are in G. P.

10. (1). How many *distinct* throws can be made with a pair of *similar* dice?

(2). What is the sum of the points of *all* the distinct throws?

11. If $t_{r-1} . t_r . t_{r+1}$ be 3 *successive* terms of an expanded binomial, show that $t_r^2 : t_{r-1} . t_{r+1} = (r+1)(n-r+1) : r(n-r)$.

L

1. Define the terms "like quantities" and "a prime number."

Find the value of—

$$10a^3b + \frac{a^2+b^3}{c} - ab^2\left\{(5c-c^2b)(7ac+8b^2+5bc^2) - \sqrt{(3ab)}\right\} + \frac{a-b}{a^2+b-c}$$

when $a = -3$, $b = 4$, and $c = \frac{1}{2}$.

And reduce to its lowest terms the expression

$$\frac{1+x^{\frac{1}{2}}+x+x^{\frac{3}{2}}}{2x+2x^{\frac{3}{2}}+3x^2+3x^{\frac{5}{2}}}$$

2. Simplify $\dfrac{8-5\sqrt{(2)}}{3-2\sqrt{(2)}}$.

Find the square root of $28+10\sqrt{(3)}$.

3. Show that a "simple equation" can have only one solution. Solve

(i) $\dfrac{(2x+3)x}{2x+1} + \dfrac{1}{3x} = x+1$.

(ii). $\left.\begin{array}{l} 3x+5y=8 \\ 4x+3y=7 \end{array}\right\}$

(iii). $x^2 - \dfrac{3}{4x} = 1\frac{9}{16}$.

4. Find a number such, that whether it be divided into *three* or *four* equal parts, the continued product will be the same.

The rent of a village is paid in certain fixed numbers of maunds of wheat and sugar. When wheat is at Rs 2 per maund, and sugar at Rs 5 per maund, the portions of rent by wheat and sugar are equal to one another; but when wheat is at Rs 2-8 a maund, and sugar at Rs 6 a maund, the rent is increased by Rs 180. What is the rent in produce?

5. If $a : b :: c : d$, prove that

$$a+b : a-b :: c+d : c-d.$$

The 15th term of a series 9, $14\frac{1}{3}$, $19\frac{2}{3}$, &c., is equal to the 5th term in a Geometrical progression, beginning with unity. Find the Common Ratio of the Geometrical progression.

Find the number of shot in a pile, whose base is a square, each side containing 20.

6. A starts from M at the same time that B starts from R, towards Dehra. After A had overtaken B, they found that the distance A had travelled together with the distance B had travelled, made up 90 miles; that A had passed through R 5 hours before; and that B at his rate of travelling was 20 hours journey from M. Required the distance between the two places.

7. How many different sums can be made up of a rupee, an eight anna piece, a four anna piece, a pice, and a pie?

Find the 7th term of the expression $(4x+3y)^{10}$.

M.

1. Given $\dfrac{3x+2y\sqrt{(-1)}}{5\sqrt{(-1)}-2} = \dfrac{15}{8x+3y\sqrt{(-1)}}$,

find real values of x, y.

2. Solve $3x + \sqrt{(30x-71)} = 5$.

3. In what year next after this date will five Sundays fall in February?

4. The product of two numbers is p, and the difference of their cubes is equal to m times the cube of their difference. Find the numbers.

5. From a vessel containing a gallons of wine, b gallons are drawn off, and the vessel filled up with water. Find the quantity of wine left after n such operations.

6. Show that $\dfrac{n(n^2+6n+11)}{6}$

is an integer, when n is an integer.

7. Find the series in Arithmetical Progression the sum of the first n terms of which is equal to n^2 whatever be the value of n.

8. Show that every coefficient in the expansion of $(a+b)^n$ is an integer, when n is an integer.

9. Find the value of n in the binomial $(a+b)^n$, all the coefficients in whose expansion are equal in magnitude, and are alternately $\pm$.

10. n points are taken on a plane, so that p circles having no common points can be drawn, each passing through four points $\left(p \text{ being} < \dfrac{n}{4}\right)$. How many different circles can be drawn, each passing through as many points as possible?

11. Prove the truth of the test of arithmetical multiplication, called, "Casting out the nines".

N

1. Add $\dfrac{2}{a-b}+\dfrac{2}{b-c}+\dfrac{c}{c-a}+\dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}$

Multiply $\dfrac{x^2+xy}{x^2+y^2}$ by $\dfrac{x^3-y^3}{xy(x+y)}$.

2. Solve the following equations :—

$$(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2\ ;$$

$$\frac{m(x+a)}{x+b}+\frac{n(x+b)}{x+a}=m+n.$$

3. Solve

$$\left.\begin{aligned} 13x+11y&=4a \\ 12x-6y&=a \end{aligned}\right\}$$

$$\sqrt{(x^2+9)}+\sqrt{(x^2-9)}=\sqrt{(34)}+4.$$

4. 2 plugs are opened in the bottom of a cistern containing 192 gallons of water ; after 3 hours, one of the plugs becomes stopped, and the cistern is emptied by the other in 11 more hours : had the stoppage occurred after 6 hours, only 6 hours more would have been required to empty the cistern.

How many gallons will each plug hole discharge in an hour, supposing the discharge uniform.

5. Form the quadratic equation whose roots are 4 and 5.

6. Find the sum of $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+$ &c., to infinity.

7. There are n points in a plane, no three of which are in the same straight line with the exception of p, which are all in the same straight line.

(1) Find the number of straight lines which result from joining them.

(2) Find the number of triangles which can be formed by joining them.

8. Expand $(x-a)^6$ by the binomial theorem.

Prove that in the expansion of $(1+x)^n$ the coefficient of the r^{th} term from the beginning, is equal to the coefficient of the r^{th} term from the end.

O

1. Extract the square root of $13-42\sqrt{(-4)}$.

2. Solve the equations :—

$$(a)\quad (1+x)^{\frac{2}{m}}-3(1-x)^{\frac{2}{m}}=2(1-x^2)^{\frac{1}{m}}$$

$$(\beta)\quad \left.\begin{array}{ll} x^2+\tfrac{1}{2}x+y^2+\tfrac{1}{2}y & =32 \\ xy & =14 \end{array}\right\}$$

$$(\gamma)\quad \left.\begin{array}{ll} w+x+y+z & = 44 \\ wx+yz & =250 \\ wy+xz & =234 \\ wz+xy & =225 \end{array}\right\}$$

3. M starts in a Velocipede from Roorkee to Meerut, and N at the same time in a Dâk Gharry from Meerut to Roorkee, and they travel uniformly. M reaches Meerut in 4 hours, and N reaches Roorkee 9 hours after they have met on the road. Find in what time each has performed the journey.

4. A travels 12 miles the first day, and increases his rate of travelling by 2 miles daily; B sets off from the same place 2 days after, and travels 30 miles daily on the same road. How long after B's departure are A and B at the same point of the road.

5. The combinations of n articles taken 2 together, are to those taken 4 together as $1 : 7\tfrac{1}{2}$. Find n.

6. Sum the series—

$9+11+15+23+\&c.$, to n terms ; also to 8 terms.

7. What are the first five terms of the expansion of $\sqrt[3]{(a+x)}$ in a series, by the Binomial Theorm ?

P

1. Express $\dfrac{a+b\sqrt{(-1)}}{c+d\sqrt{(-1)}}$ in the form $A+B\sqrt{(+1)}$.

2. Extract the cube root of—

$$\left\{x+(a^2x)^{\frac{1}{3}}\right\}^2+\left\{a+(ax^2)^{\frac{1}{3}}\right\}^2 \text{ and of } -11-2\sqrt{(-1)}.$$

3. Find the Least Common Multiple of—

$$x^2-y^2,\ 3(x-y)^2 \text{ and } 12\,(x^3+y^3).$$

4. Solve the following equations—

(a). $(a+x)^{\frac{1}{m}}=(x^2+8ax+b^2)^{\frac{1}{2m}}$ for x.

(β). $\dfrac{x}{a+x}+\dfrac{a}{(a+x)^{\frac{1}{2}}}=\dfrac{b}{x}$ ....for x.

(γ). $\left.\begin{array}{l} x+y+z\quad =11 \\ x^2+y^2+z^2=49 \\ yz\qquad\quad =3x(z-y) \end{array}\right\}$ for x, y and z.

5. A gentleman sends a lad into the market to buy a shilling's worth of oranges. The lad eats two. The gentleman pays at the rate of one penny for 15 more than the market price. How many did the gentleman get for his shilling ?

6. If S_a represent the sum of n of the natural numbers beginning with a, prove that $S_{3a+n-1}=3S_a$.

And insert 6 Arithmetical means between 1 and 29.

7. Sum to n terms $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+$ &c.

8. A boat's crew consists of 8 men, 3 of whom can only row on one side, and 2 only on the other side. Find the number of ways in which the crew can be arranged.

9. Find the sum of the coefficients of the terms in the expansion $(1+x)^n$. And write down the coefficient of (y) in the expansion of

$$\left(y^2+\frac{c^3}{y}\right)^5.$$

11. Write down the coefficient of x^{2r+1} in the expansion of

$$\left(x-\frac{1}{x}\right)^{2n+1}.$$

Q

1. Simplify $$\frac{(ay-bx)^2+(ax+by)^2}{\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{x}{y}+\frac{y}{x}\right)};$$
and divide $1+10x^3+27x^6$ by $1-2x+3x^2$.

2. If four numbers are proportionals, prove that

(1) Their reciprocals are proportionals;

(2). The greatest and least of them together are greater than the other two.

3. How many different arrangements can there be of n letters, a, b, c, ... ?

In how many of these arrangements will a and b be next to one another? In how many of them will a come before b (but not necessarily immediately before b)?

4. Sum the Series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+$ ......to n terms; and show that the sum of any odd number of terms of this series is always greater, the sum of any even number of terms always less, than the sum to infinity.

What is the least number of terms of the series which will give a sum differing from the sum to infinity by less than ·0001?

5. Solve the equations

$$(\alpha)\ldots\frac{x-1}{x+1}+\frac{x+1}{x-1}=\frac{5x}{x^2-1};$$

$$(\beta)\ldots\begin{cases}\dfrac{x}{a+2}+\dfrac{y}{a}=1,\\[2mm]\dfrac{x}{a}+\dfrac{y}{a-1}=1.\end{cases}$$

Is there any value of a for which the equations (β) are not resoluble?

6. An annuity of £P per annum is to begin n years hence, and is to be payable for ever. Find its present value at r per cent. rate of interest; and show that its present value is to its value n years hence as $(1+r)^{-n}:1$.

7. What is meant by the base of a system of Logarithms? what are the advantages of taking 10 as the base?

Prove that if $N=\frac{P}{Q}$, $\log N=\log P-\log Q$; and find, as far as four places of decimals, the number of which the logarithm to base 10 is ·5.

[R]

1. For a house occupied by B, A pays a rent of £40 per annum by equal payments at the end of each quarter. B pays A by equal payments in advance at the beginning of each month. How much a month ought B to pay in order that at the end of the year, with simple interest reckoned at $3\frac{1}{3}$ per cent. per annum, A may have recovered the value of his own four payments with one-tenth additional?

2. Shew how to find the lowest common multiple of three algebraical expressions.

If l_1, l_2, l_3 are the lowest common multiples of B and C, of C and A, of A and B, respectively; if g_1, g_2, g_3 are the highest common divisors of the same pairs; and if L, G are the lowest common multiple and highest common divisors of A, B and C:

$$\text{prove that } \frac{L}{G}=\left(\frac{l_1 l_2 l_3}{g_1 g_2 g_3}\right)^{\frac{1}{2}}.$$

3. (a) Find the simplest expression for

$$\frac{(a+p)(a+q)}{(a-b)(a-c)(a+x)}+\frac{(b+p)(b+q)}{(b-c)(b-a)(b+x)}+\frac{(c+p)(c+q)}{(c-a)(c-b)(c+x)}.$$

(β) If the letters all denote positive quantities, prove that

$$\frac{(a+b)xy}{ay+bx} \text{ is never greater than } \frac{ax+by}{a+b}.$$

4. Find in terms of the coefficients the sum and product of the roots of the equation.

$$ax^2+2bx+c=0.$$

Find the condition that the roots of $ax^2+2bx+c=0$ may be formed from those of $a'x^2+2b'x+c'=0$ by adding the same quantity to each root.

5. Solve the equations,

$$(a)\quad (c+a-2b)x^2+(a+b-2c)x+(b+c-2a)=0.$$

$$(\beta)\quad ax-yz=ay+zx=az+xy=b^2.$$

6. Investigate the formula for the number of combinations of n things taken r at a time, without assuming the formula for permutations.

A selection of c things is to be made, part from a group of a things and the remainder from a group of b things. Prove that the number of ways in which such a selection may be made will never be greater than when the number of things taken from the group of a things is the integer next less than $\dfrac{(a+1)(c+1)}{a+b+2}$.

7. Shew that corresponding small increments of a number and its logarithm are proportional.

Find n from the follwoing data :

$$\log_{10}42563=4{\cdot}6290322,$$
$$\log_{10}42564=4{\cdot}6290424,$$
$$\log_{10}n \quad =2{\cdot}6290376.$$

Find to how many decimal places n can be determined by this method, given that $\log_{10}e={\cdot}43429$.

APPENDIX.

MISCELLANEOUS EXAMPLES.

[FROM THE LILAVATI AND VIJA GANITA OF BHASKARA ACHARYA.]

1. What is that number, which multiplied by five, and having the third part of the product subtracted, and the remainder divided by ten, and one-third, a half and a quarter of the original quantity added, gives two less than seventy ?

2. Out of a heap of pure lotus flowers, a third part, a fifth and a sixth, were offered respectively to the gods Siva, Vishnu and the Sun ; and a quarter was presented to Bhavani. The remaining six lotuses were given to the venerable preceptor. Tell quickly the whole number of flowers.

3. A traveller, engaged in a pilgrimage, gave half his money at Prayága ; two-ninths of the remainder at Cási ; a quarter of the remainder in paying of taxes on the road ; six-tenths of what was left at Gayá ; there remained sixty-three *nishcas* (a *nishca* is, a gold mohur) with which he returned home. Tell me the amount of his original stock of money, if you have learned the method of reduction of fractions of residues.

4. Out of a swarm of bees one-fifth settled on a blossom of *Cadamba* ; and one-third on a flower of *Silindhri* ; three times the difference of those numbers flew to the bloom of *Cutaja*. One bee, which remained, hovered and flew about in the air, allured at the same moment by the pleasing fragrance of a jasmin and pandanus. Tell me the number of bees.

5. Tell me quickly, skillful calculator, what numbers are they, of which the difference is eight, and the difference of squares four hundred ?

6. One pair out of a flock of geese remained sporting in the water, and saw seven times the half of the square root of the flock proceeding to the shore tired of the diversion. Tell me, dear girl, the number of the flock.

7. Pártha (Arjun), irritated in fight, shot a quiver of arrows to slay Carna. With half his arrows, he parried those of his antagonist; with four times the square root of the quiverful, he

killed his horses; with six arrows, he slew Shalya (the charioteer of Carna); with three he demolished the Umbrela, standard and bow; and with one, he cut off the head of the foe. How many were the arrows which Arjun let fly?

8. The square root of half the number of a swarm of bees is gone to a shrub of jasmin; and so are eight-ninths of the whole swarm a female is buzzing to one remaining male that is humming within a lotus, in which he is confined, having been allured to it by its fragrance at night. Tell me the number of bees.

9. Tell me the number which added to its third part and eighteen times its square root, amounts to twelve hundred.

10. The sum of 94 *niśchas* being lent in three portions at 5, 3 and 4 per cent. interest an equal interest was obtained on all three portions, in seven, ten, and five months respectively. Tell mathematician, the number of each portion.

11. If a bamboo, measuring thirty-two cubits and standing upon level ground, be broken in a place, by the force of the wind, and the tip of it meet the ground at a distance of sixteen cubits from its root; say mathematician at what distance from the root was it broken?

12. A snake's hole is at the foot of a pillar 9 cubits high, and a peacock is perched on its summit. Seeing a snake, at a distance of thrice the pillar, gliding towards his hole, he pounces obliquely upon him. Say quickly at how many cubits from the snake's hole do they meet, both proceeding an equal distance.

13. In a certain lake swarming with ruddy geese and cranes, the tip of a bud of lotus was seen a span above the surface of the water. Forced by the wind, it gradually advanced, and was submurged at the distance of two cubits. Compute quickly, mathematician, the depth of water.

14. The hypotenuse of a right angled triangle is 17 cubits; and the sum of the base and perpendicular is 23, tell them to me.

15. The hypotenuse of a right angled triangle is 13 cubits, and the difference of the side and the perpendicular is 7 cubits; what are the side and perpendicular?

16. Tell the perpendicular drawn from the intersection of strings stretched mutually from the roots to the summits of two bamboos 15 and 10 cubits high upon ground of unknown extent.

17. Four jewellers, possessing respectively eight rubies, ten sapphires, a hundred pearls and five diamonds, presented each from his own stock, one piece to the others in token of regard and gratification at meeting; and they thus became owners of precisely equal value of 932 *Nishcas*. Tell, friend, what were the prices of their gems respectively?

18. Tell me, friend, the perpendicular in a triangle in which the base is fourteen, one side fifteen, and the other thirteen.

19. In an A. P., $\frac{1}{3}(n-1)$ is the first term, $\frac{1}{4}(n-1)$ is the common difference; the sum is equal to the product of the first term, common difference and the number of terms together augmented by the addition of the seventh part of the product. Find the first term, common difference and the number of terms.

20. What is the number, learned man, which being multiplied by 12 and added to the cube of the number, is equal to six times the square added to 35?

21. The square of the eighth part of a troop of monkeys was skipping in a forest being delighted with sport; the 12 remaining were seen playing on a hill, chattering to each other. How many were they in all?

22. The fifth part of a troop of monkeys less three squared, had gone to a cave; the remaining one was seen climbing on the branch of a tree. Say how many were they?

23. Tell me, friend, the base, perpendicular and hypotenuse of a right angled triangle, in which the square-root of the base diminished by three, being lessened by one, is the difference, between the perpendicular and hypotenuse; the perpendicular is equal to the square of half the base diminished by one.

24. The sum of the base, perpendicular and hypotenuse is 40 and the product of the perpendicular and base one hundred and twenty. What are they?

25. The sum of the base, perpendicular and hypotenuse is 56 and their product is seven times six hundred. What are they?

26. What number is it, which being divided by six, has five for a remainder, or divided by five, has four for a remainder or divided by four, leaves three, for a remainder, or divided by 3, leaves two? The product of the first and fourth quotients exceeds that of the second and third by 17.

27. Find the numbers which being multiplied by 5, 7 and 9 and the products divided by 20, leave remainders increasing in progression by the common difference one, and quotients equal to the remainders ; *the square of the excess of the second over the third is less than the third by 3.*

28. What number, being divided by two, has one for a remainder; and divided by three, has two; and divided by five has, three; and the quotients also, like itself; of the second quotients, the square of the second exceeds the product of the first and third by 50.

29. A horse was purchased, with the principal sums, one two &c. up to nine, by dealers in partnership; it was sold by them for 495; what did each gain?

30. The 1st term of an *A.P.* is 7, the common difference 5, the period or number of terms eight, tell me, what are the middle and last terms? And what is the total sum?

31. A person gave three *drammas* on the first day, and continued to distribute alms increasing by two a day; and he thus bestowed on the priests three hundred and sixty *drammas*; say quickly in how many days? (*dramma*=a Rupee.)

32. The first term of a G. P. is two, the common increase three-fold and the number of terms 7. Find the sum.

33. In a pleasant, spacious and elegant edifice, with eight doors, constructed by a skillful architect as a palace for the lord of the land, tell me the permutations of aperture taken one, two three &c. (at a time).

34. Say mathematician, how many are the different tastes, sweet, pungent, astringent, sour, salt and bitter, taken them by ones, twos, or threes, &c. at a time.

35. How many are the variations of number with any digits except cipher exchanged in six places of figures?

36. How many are the variations of form of the god Sambhu by the exchange of his ten attributes held reciprocally in his several hands; namely the rope, the elephant's hook, the serpent, the tabor, the skull, the trident, the bedstead, the dagger, the arrow, and the bow. Likewise of god Hari by the exchange of his four attributes held reciprocally in his four hands.—namely, conch, the discus, the mace and the lotus.

ANSWERS.

PART II.

Ex. LXV. p. 3.

1. $x = \pm 4$. 2. ± 2. 3. ± 2. 4. ± 2.
5. $\pm \frac{1}{2}$. 6. ± 2. 7. ± 2. 8. ± 4.
9. ± 6. 10. $\pm a\sqrt{3}$. 11. $\pm \sqrt{\frac{4}{5}}$. 12. $\pm \frac{1}{2}\sqrt{3}$.
13. ± 1. 14. $\pm a$. 15. $\pm \frac{1}{2}\sqrt{5}$. 16. $\pm \frac{1}{2}a\sqrt{5}$
17. ± 1. 18. $\pm \sqrt{2}$. 19. $\pm \dfrac{ab}{\sqrt[2n]{(b^{2n}-1)}}$.

20. $\pm \dfrac{a}{\sqrt{\left(a^{\frac{2n}{n+1}} b^{\frac{2n}{n+1}} - 1\right)}}$. 21. $\pm \frac{5}{3}a$. 22. $\pm a\sqrt{\dfrac{2b}{b^2+1}}$.

Ex. LXVI. p. 13.

1. $x = 1$, or 2. 2. 2, 3. 3. 1, 6. 4. 2, $\frac{1}{2}$.
5. $\frac{1}{2}$, $\frac{1}{3}$. 6. $\frac{3}{2}$, $\frac{2}{3}$. 7. $\frac{1}{2}$, -3. 8. $-\frac{2}{3}$, $-\frac{3}{2}$.
9. $\frac{1}{2}$, $-\frac{2}{3}$. 10. $\frac{4}{3}$, $-\frac{1}{3}$. 11. 7, $-\frac{10}{3}$. 12. 5, $-\frac{17}{3}$.
13. 5, $-\frac{13}{3}$. 14. 5, 3. 15. $\frac{1}{3}$, $-\frac{4}{3}$. 16. $\frac{4}{3}$, -2.
17. 12, -2. 18. 2, $\frac{1}{3}$. 19. 2, 5. 20. a, $-b$.
21. $a \pm b$. 22. 10, -110. 23. 5, $-\frac{7}{3}$. 24. 4, $\frac{8}{7}$.
25. 3, $-\frac{3}{13}$. 26. 2, $-\frac{41}{24}$. 27. $a+1$, $\dfrac{1}{a+1}$.
28. 6, $\frac{49}{13}$. 29. 2, $-\frac{81}{10}$. 30. 2, $-\frac{14}{9}$. 31. 2, 6.
32. 3, -1. 33. 3, 5. 34. 1, $\dfrac{2b}{a-b}$. 35. 5, $\frac{1}{2}$.
36. -5, $-\frac{1}{2}$. 37. $\frac{3}{2}$, $\frac{1}{2}$. 38. a, b. 39. a, $-b$.
40. a^3, a^2. 41. $a \pm n$. 42. $\dfrac{2}{a}$, $\dfrac{1}{a}$. 43. $\dfrac{b}{c}$, $\dfrac{b}{a}$.
44. $a-b$, $\dfrac{b}{2}$. 45. $\dfrac{a+\sqrt{(ab)}+b}{a-\sqrt{(ab)}+b}$, $\dfrac{a-\sqrt{(ab)}+b}{a+\sqrt{(ab)}+b}$.
46. $(a \pm b)^2$. 47. 1 or $\dfrac{2a}{b-a}$. 48. $a-1$, $\frac{1}{2}$.

49. $b, -\dfrac{a^2+ab}{a+2b}$ 50. $\dfrac{a}{1-a}, -\dfrac{a}{1+a}$.

51. $\dfrac{b-c+a^2}{a}, \dfrac{b-c-a^2}{a}$. 52. $a+b+c, a+b-c$.

53. $a+c, 2b$. 54. $\dfrac{2a}{a-b}, \dfrac{2b}{a-b}$.

55. $2(2+\sqrt{3}), 2+\sqrt{3}$. 56. $\frac{2}{7}(3+\sqrt{2}), -\frac{2}{7}(3+\sqrt{2})$.

57. $\sqrt{3}, \dfrac{1}{\sqrt{3}}$. 58. 4, 16. 59. $\dfrac{a+b}{c}, -1$.

60. $2+\sqrt{3}, -4$. 61. $4, \frac{1}{4}$.

62. $\dfrac{a}{n}\{1 \pm \sqrt{(n^2+1)}\}$. 63. $8, -5$.

64. $a, -9a$. 65. $2, -3$. 66. $3, -\frac{10}{3}$.

67. $\dfrac{a^2}{b}, -\dfrac{3a^2}{5b}$. 68. $5, -3$. 69. $4, -\frac{22}{5}$.

70. $9, 3\frac{3}{5}$. 71. $5, \frac{7}{10}$. 72. 3, 4.

73. $-13, 3$. 74. $2, -\frac{3}{2}$. 75. $3a, -4a$.

76. $\frac{2}{3}a(-1 \pm \sqrt{7})$. 77. $0, a$. 78. $\frac{2}{5}\{6 \pm \sqrt{(11)}\}$.

79. $0, 2a$. 80. $\dfrac{na}{2}, \dfrac{a}{2n}$. 81. $-a, \frac{3}{5}a$.

82. $1, \frac{3}{5}$. 83. $0, \frac{2}{5}a$. 84. $a, \frac{1}{5}a$.

Ex. LXVII. p. 28.

1. $x = \pm 2, \pm 1$. 2. $2, 1, -1 \pm \sqrt{(-3)}, -\frac{1}{2}\{1 \pm \sqrt{(-3)}\}$.
3. 1, 2, &c. 4. 4, 1. 5. 81, 25.
6. $\sqrt[3]{(\frac{1}{2}2)^2}, 4$. 7. $x^m = 4, 1$. 8. $\pm 4, \pm 5$.
9. $25a, 4a$. 10. $\pm 5, \pm \sqrt{(-\frac{47}{2})}$. 11. $\cdot 64, -1331$.
12. $1, 7^n$. 13. $8, \frac{1}{27}$ 14. $32, \frac{1}{32}$.
15. 11, 3. 16. $2, -\frac{3}{4}$. 17. $x^n = 16, 1$.
18. $x^n = \frac{1}{2}, -1$. 19. $16, \frac{1}{16}$. 20. $x^n = 16$ or $\frac{1}{16}$.
21. $\pm 5, \pm \sqrt{(10)}$. 22. $\pm \dfrac{11}{2\sqrt{2}}, \pm \sqrt{\frac{41}{8}}$. 23. $\pm 3, \pm \sqrt{(618)}$.
24. $3, -4, -\frac{1}{2}\{1 \pm \sqrt{(129)}\}$. 25. $3, -\frac{1}{2}, \frac{1}{2}, 2$.
26. $4, 2, 1, \frac{1}{2}$. 27. $6, -1, \frac{1}{2}\{5 \pm \sqrt{(-39)}\}$.
28. $8a, -a, \frac{1}{2}a\{7 \pm \sqrt{(-71)}\}$. 29. $a, 2a, 3a, 4a$.

30. $2a, 3a, \frac{1}{2}a, \frac{1}{3}a.$ 31. $\frac{1}{4}, \frac{1}{16}, 4, 16.$

32. $4a, -\frac{8}{3}a, \frac{1}{3}a\{2 \pm \sqrt{(67)}\}.$

33. $a+2, -\frac{1}{3}(a+6), \frac{1}{3}\{a \pm 2\sqrt{(a^2-3a)}\}.$ 34. $0, \pm\sqrt{(\frac{1}{2} \pm \frac{1}{2}\sqrt{5})}.$

35. $x=2$ or $3^x=-12.$ 36. $\pm 5, \pm 4\sqrt{2}.$

37. Divide by $x^{\frac{1}{r}}$ etc.; a quadratic in $x^{\frac{r-p}{2pr}}.$

38. $4, \frac{4}{25}.$ 39. $2, \frac{1}{2}, \frac{1}{4}\{-7 \pm \sqrt{(33)}\}.$

40. A quadratic in $x-\sqrt{(x)}.$

41. $3, -2, \frac{1}{2}\{1 \pm \sqrt{(21)}\}.$ 42. $3, -\frac{1}{3}, \frac{1}{4}\{5 \pm \sqrt{(1329)}\}.$

43. Multiply by $\frac{25}{116} \times (x^2+x+5)$ and transpose, then $(x^2+x+5)-\frac{125}{116}x\sqrt{(x^2+x+5)}-\frac{25}{116}x^2=0.$ Hence $\sqrt{(x^2+x+5)}=\frac{125}{232} \pm \frac{165}{232}x.$ $\therefore x=4, -\frac{20}{9},$ etc.

44. $4, \frac{1}{4}, 17 \pm 12\sqrt{2}.$

45. $$\frac{5x^2+2x}{(x^2+2x-2)(x^2-2x+3)}=\frac{x}{2}, \therefore x=0, \pm\sqrt{5}, \pm\sqrt{(-2)}.$$

46. $0, 1, 2.$ 47. $2, -1 \pm \sqrt{(-1)}.$ 48. $25, 529.$

49. $64, 9.$ 50. $x=5$ one solution, also $(x^2-8x+7)(x-3) = (x^2-8x+12)(x-4) = (x^2-8x+7+5)(x-4) = (x^2-8x+7)(x-4)+5(x-4).$ $\therefore (x^2-8x+7)(x-3-x+4)=5x-20.$ $\therefore x=\frac{1}{2}\{13 \pm \sqrt{(61)}\}.$

51. $x=5$ or 70; the latter value satisfies only the equation $-\sqrt[4]{(x+11)}+\sqrt{(x+11)}=6.$

52. $15, -27,$ etc. 53. $a, \frac{1}{3}a, -\frac{1}{3}a.$

54. $0, 17, \frac{17}{2} \pm \sqrt{\frac{141}{2}}.$ 55. $1, 4, \pm\sqrt{(-1)}.$

56. $1, 2, -\frac{1}{2}\{5 \pm \sqrt{(17)}\}.$ 57. $-1, 3, \underline{1 \pm 5\sqrt{3}}.$ It should be $1 \mp 2\sqrt{19}$

58. $13\frac{7}{16}, 78\frac{7}{16}$; the latter value satisfies only the equation $\sqrt{(\frac{41}{16}+x)}-\sqrt[4]{(\frac{41}{16}+x)}=6.$

59. Let $\sqrt[4]{(\frac{41}{16}+x)}=a$ and $\sqrt[4]{(\frac{41}{16}-x)}=b$; then $a-b=1.$ $\therefore a^4+b^4-4ab(a-b)^2-2a^2b^2=1.$ $\therefore \frac{41}{8}-4ab-2a^2b^2=1.$ Hence $ab=\frac{3}{4}$ or $-\frac{11}{4}.$ $\therefore (\frac{41}{16})^2-x^2 = (\frac{9}{16})^2$ or $(\frac{121}{16})^2$; $\therefore x=\pm\frac{5}{2}, \pm\frac{1}{2}\sqrt{\left(\frac{-405}{2}\right)}.$

60. $\dfrac{8(x-1)^2}{(4x-5)(4x-3)(x-1)^2}=8(x-1)^2$. $\therefore\ x=1$ one solution also $(16x^2-32x+15)(x^2-2x+1)=1$. $\therefore\ 16(x^2-2x+1)^2-(x^2-2x+1)=1$, &c.

61. $\dfrac{4}{(x-3)(x-1)}-\dfrac{2}{(x-2)(x-1)}=(x-2)(x-1)$.

$\therefore\ \dfrac{2(x-1)}{(x-2)(x-3)}=(x-2)(x-1)^2$.

$\therefore\ x=1$, one solution; also $(x^2-4x+4)(x^2-4x+3)=0$;

$\therefore\ (x^2-4x+4)^2-(x^2-4x+4)=2$; &c.

$\therefore\ x=2\pm\sqrt{2},\ 2\pm\sqrt{(-1)}$ &c.

62. $(4x-1)(4x-3)(2x-1)^2=2$; $\therefore\ \{4(4x^2-4x+1)-1\}\{4x-4x+1\}=2$; etc.

63. $2,\ 6,\ 4\pm\sqrt{(11)}$. 64. $1, 3, 2\pm\sqrt{(-7)}$. 65. $5, -1$.

66. $\sqrt{(a+b)}\{\sqrt{(a+x)}+\sqrt{(b-x)}\}=\sqrt{\{(a+x)(b-x)\}}$; squaring we get a quadratic in $\sqrt{\{(a+x)(b-x)\}}$, etc.

67. $4, -3, \frac{1}{2}\{1\pm\sqrt{(-43)}\}$. 68. $\frac{1}{2}, 2, \frac{1}{4}\{-13\pm\sqrt{(153)}\}$.

69. $16, 9$. 70. $1, -8, 10$. 71. $1, 4$.

72. $6, -7, \frac{1}{2}\{-1\pm\sqrt{(221)}\}$. 73. $5, -\frac{13}{2}, \frac{1}{4}\{-3\pm\sqrt{(794)}\}$.

74. $1, 4$. 75. $1, 5$. 76. $3, -\frac{1}{2}, \frac{1}{4}\{25\pm\sqrt{(521)}\}$.

77. $\frac{3}{2}, \frac{1}{2}\{3\pm\sqrt{(10)}\}$. 78. $1, -\frac{1}{12}$, etc.

79. $\frac{1}{2}a\{5\pm\sqrt{(13)}\}, \frac{1}{2}a\{5\pm\sqrt{(-3)}\}$. 80. $\frac{1}{2}, -\frac{1}{12}$, etc.

81. $0, 2(1\pm\sqrt{2})$. 82. $0, -1, \frac{24}{25}$.

83. $(x^2-7x)-5=\pm\sqrt{(145)}$, etc. 84. $a, a+2\pm\sqrt{(8a+3)}$;

85. $\sqrt{x}=\frac{1}{3}(1-a)\pm\frac{2}{3}\sqrt{(a^2-2a+4)}$.

86. $2, \frac{4}{3}, \frac{1}{3}\{-5\pm\sqrt{(-14)}\}$. 87. $0, 5, \frac{5}{2}\pm\frac{1}{2}\sqrt{(-13)}$.

88. $4, \sqrt{\{2\pm\sqrt{(-11)}\}}$. 89. $\pm\sqrt{3}, \pm\sqrt{2}$.

90. $1, 2, 3, 4$. 91. $1, 1\pm\sqrt{(10)}$.

92. $x^2=-ab\pm\frac{1}{2}(a^2-b^2)\sqrt{3}$. 93. $0, \pm\frac{1}{2}\sqrt{3}$.

94. $\pm\sqrt{\{2a\pm a\sqrt{7}\}}$. 95. $x^2-5ax=5a^2$ or $-\frac{13}{3}a^2$.

96. $\pm a, \pm 2a$. 97. $\pm\frac{5}{4}a, \pm\frac{5}{3}a$.

98. $2\frac{1}{4}, \frac{1}{4}$. 99. $x=\frac{1}{2}$ is the only solution.

100. $3, \frac{1}{2}\{-3\pm\sqrt{(-15)}\}$. 101. $a, 2a, \dfrac{961}{1025}a$.

102. $a, b, c.$ 103. $\pm(m-n), \pm(m+n).$

104. $2, \pm 2\sqrt{(-1)}.$ 105. $2.$ 106. $-1, 0.$

107. $2a, \frac{7}{4}a.$ 108. $x=1$, also $x^2-6x+3=\pm\sqrt{(15)}.$

109. $3, \frac{1}{3}, 2, \frac{1}{2}, 0.$

110. $x=\frac{a}{b}, \frac{b}{a}, \frac{2a}{b}, \frac{b}{2a}.$ 111. $0, \pm\sqrt{(-5)}, \pm\sqrt{\frac{14}{3}}.$

112. $0, \pm\sqrt{\pm 5}.$ 113. $x^2-5ax=\pm\sqrt{(41)}-3.$

114. An equation in $\sqrt{(1-x^2)}.$ 115. $\pm 1, \pm 6.$

116. $\pm\sqrt{\left\{\frac{a^2}{2}\pm\frac{a(1+a^2)}{2\sqrt{(2+a^2)}}\right\}}$ 117. $\pm\frac{1}{3}\sqrt{3}, \pm\frac{1}{2}\sqrt{(-5)}.$

118. $x^4=\frac{1}{2}.$ 119. $-\frac{1}{2}a, -2a,$ etc. 120. $a,$ etc.

Ex. LXVIII. p. 44.

1. $x=\pm 3, y=\pm 2.$ 2. $x=4$ or $3, y=3$ or $4.$

3. $x=-4$ or $9, y=-9$ or $4.$ 4. $x=4$ or $3, y=3$ or $4.$

5. $x=3$ or $\frac{27}{26}, y=2$ or $\frac{86}{26}.$ 6. $x=5$ or $-\frac{41}{3}, y=4$ or $\frac{49}{3}.$

7. $x=\pm 4$ or $\pm 7\sqrt{2}, y=\pm 3,$ or $\mp 5\sqrt{2}.$

8. $x=2$ or $3, y=3$ or $2.$ 9. $x=2$ or $\frac{5}{2}, y=3$ or $\frac{5}{2}.$

10. $x=6$ or $\frac{50}{3}, y=5$ or $-16\frac{1}{3}.$ 11. $x=\pm 5, y=\pm 4.$

12. $x=\pm 3, y=\pm 1.$ 13. $x=\pm 5$ or $\pm 7, y=\pm 3$ or $\pm 1.$

14. $x=\pm 5$ or $\pm\sqrt{3}, y=\pm 4$ or $\pm 3\sqrt{3}.$

15. $x=\pm 2$ or $\pm 4, y=\pm 3.$

16. $x=\pm 5$ or $\pm 9\sqrt{3}, y=\pm 2$ or $\pm\frac{11}{3}\sqrt{3}.$ 17. $x=\pm 3, y=\pm 2$

18. $x=10$ or $-\frac{1}{10}$; $y=\frac{111}{10}$ or $1.$ 19. $x=\pm 5, y=\pm 1$

20. $x=\pm 4$ or $\pm\frac{28}{13}, y=\pm 5$ or $\pm\frac{9}{13}.$

21. $x=\pm 3$ or $\pm\sqrt{\frac{5}{13}}, y=\pm 1$ or $\mp 9\sqrt{\frac{5}{13}}.$

22. $x=4$ or $12, y=12$ or $4.$ 23. $x=3$ or $25, y=3$ or $-19.$

24. $x=5$ or $\frac{37}{35}, y=2$ or $\frac{122}{35}.$ 25. $x=\pm 5, y=\pm 1.$

26. $x=\pm\frac{15}{4}, y=\pm\frac{9}{4}.$ 27. $x=\frac{1}{2}$ or $5, y=\frac{1}{5}$ or $2.$

28. $x=3$ or $2, y=2$ or $3.$ 29. $x=\pm 9, y=\pm 3.$

30. $x=1$ or $4, y=-2$ or $1.$ 31. $x=4$ or $-5, y=3$ or $-4.$

32. $x=16$ or $4, y=4$ or $16.$

33. $x=0, 6, 2\pm\sqrt{5}, y=0, 3, -\frac{1}{2}\pm\frac{1}{2}\sqrt{5}.$

34. $x=\pm\sqrt{3}$ or $\pm 2\sqrt{\frac{7}{3}}, y=\pm\frac{1}{3}\sqrt{(35)}$ or $\pm 2.$

35. $x=\frac{1}{2}\{a-b\pm\sqrt{(a^2+2ab-3b^2)}\}, y=\frac{1}{2}\{a-b\mp\sqrt{(a^2+2ab-3b^2)}\}.$

36. $x=2$ or $\frac{13}{3}$, $y=1$ or $-\frac{11}{3}$. 37. $x=0$ or 2, $y=0$ or 4.

38. $x=2$ or -3, $y=3$ or -2. 39. $x=1$ or 3, $y=3$ or 1.

40. $x=\pm 3$, $y=\pm 2$. 41. $x=0$ or $\dfrac{a^2-b^2}{mb+an}$,

$$y=0 \text{ or } \frac{a^2-b^2}{ma+bn}.$$

42. $x=\pm 3$ or $\pm 5\sqrt{\frac{2}{7}}$, $y=\pm 2$ or $\mp\sqrt{\frac{2}{7}}$.

43. $x=\frac{1}{2}$ or 2, $y=6$ or 3. 44. $x=16, 9$; $y=9, 16$.

45. $x=4$ or 2, $y=2$ or 4. 46. $x=1$ or 3, $y=3$ or 1.

47. $x=4$ or -2, $y=2$ or -4. 48. $x=3$ or 4, $y=4$ or 3.

49. $x=3$ or 4, etc., $y=4$ or 3, etc.

50. $x=\pm 10$ or $\pm\sqrt{\frac{217}{2}}$, $y=\pm 6$ or $\pm\sqrt{\frac{55}{2}}$.

51. $x=1, 2$ etc., $y=\pm 2$, ± 1 or etc.

52. $x=8$ or -1, $y=1$ or -8. 53. $x=-4$ or 9, $y=-9$ or 4.

54. $x=\pm 1$, etc., $y=\pm 1$, etc. 55. $x=bc$, $y=ac$.

56. $x=\frac{3}{2}$ or $\frac{1}{6}$, $y=\frac{5}{2}$ or $\frac{5}{18}$. 57. $x=4$ or 5, $y=5$ or 4.

58. $x=16$ or 25, $y=25$ or 16. 59. $x=\pm 1$ or ± 2, etc.

60. $x=5$ or $\frac{17}{16}$, $y=3$ or $-\frac{3}{16}$. 61. $x=\pm\frac{1}{2}$, $y=\pm 3$ or $\pm\frac{1}{3}$.

62. $x=y=\sqrt[3]{5}$, etc. √63. $x=3$, $-\frac{12}{11}$, etc., $y=4$, $-\frac{29}{11}$, etc.

64. $x=2$, $\sqrt[3]{\frac{5}{2}}$, $3\sqrt[3]{\frac{1}{2}}$; $y=1$, $\sqrt[3]{\frac{5}{2}}$, $\sqrt[3]{\frac{1}{2}}$.

65. $x=4$ or 8, $y=1$ or 3. 66. $x=1$ or 25, $y=25$ or 1.

67. $x=2, 3$, or $\frac{1}{2}\{-11\pm\sqrt{(97)}\}$; $y=3, 2$ or $\frac{1}{2}\{-11\mp(97)\}$.

68. $x=\pm\dfrac{ab^2}{\sqrt{(a^4+b^4)}}$, $y=\pm\dfrac{a^2b}{\sqrt{(a^4+b^4)}}$.

69. $x=4$, -2 or $1\pm\sqrt{(-15)}$; $y=2$, -4 or $-1\pm\sqrt{(-15)}$.

70. $x=2$, -1 or $\frac{1}{2}(1\pm\sqrt{(-11)}$; $y=1$, -2 or $\frac{1}{2}(-1\pm\sqrt{(-11)}$.

71. $xy=\pm 2$ or $\pm 2\sqrt{(-1)}$, $x+y=\pm 3$, $\pm 3\sqrt{(-1)}$, etc.

72. $x=\pm 2$ or ± 1, $y=\pm 1$ or ± 2. 73. $x=\frac{1}{4}$ or 1, $y=1$ or $\frac{1}{4}$.

74. $x=9$ or $\frac{1}{4}$, $y=1$ or $\frac{25}{4}$. 75. $x=3$ or 1 etc., $y=1$ or 3 etc.

76. $x=\dfrac{a}{2}\pm a\sqrt{\dfrac{133}{20}}$, $y=-\dfrac{a}{2}\pm a\sqrt{\dfrac{133}{20}}$.

77. $x=3$, $y=1$, etc. 78. $x=3$ or 1, etc., $y=1$ or 3 etc.

79. $x=1, 2$, etc., $y=2, 1$, etc. 80. $x=1$ or 2, $y=2$ or 1.

81. $\frac{20}{3}$ or 3, etc.; $y=-10$ or 1, etc. 82. $x=2$ or 9, $y=1$ or -6.

83. $x=1$ or $\frac{19}{3}$, etc., $y=1$ or $-\frac{10}{3}$, etc. 84. $x=\pm\frac{17}{2}$, $y=\pm 2$.

85. $x=\pm 1$ or $\pm\frac{1}{2}\sqrt{(-\frac{1}{3})}$, $y=\pm\frac{2}{3}$, etc.

86 and 87. $x=\pm\frac{5}{4}$ or $\pm\frac{5}{3}$; $y=\pm\frac{5}{3}$ or $\pm\frac{5}{4}$.

88. $x=\pm 2$, $y=\pm 4$, $z=\pm 8$.

89. $x=2$ or $-\frac{12}{5}$, $y=\pm 3$ or $\pm\frac{1}{5}\sqrt{(401)}$, $z=4$ or $-\frac{24}{5}$.

90. $x=1$ or 4, $y=\pm 2$, $z=4$ or 1. 91. $x=4$ or 1, $y=\frac{1}{2}$ or 2, $z=3$ or 5. 92. $x=1, 2$ or 3; $y=2, 3$ or 1; $z=3, 1$ or 2.

93. $x=\pm(4+\sqrt{2})$, $y=\pm\sqrt{(14)}$, $z=\pm(4-\sqrt{2})$.

94. $x=2$ or $-\frac{7}{3}$, $y=\pm 3$ or $\pm\sqrt{\frac{29}{2}}$, $z=4$ or $-\frac{8}{3}$.

95. $x=1$, $y=z=\frac{5}{2}$. 96. $x=\pm 1$, $y=\pm 2$, $z=\pm 3$.

97. $x=\pm 2$, $y=\pm 4$, $z=\pm 8$, etc.

98. $x=\pm ab$, $y=\pm bc$, $z=\pm ac$.

99. $x=2$, $y=2$, $z=8$, etc. 100. $x=\pm 1$, $y=\pm 2$, $z=\pm 3$.

101. $x=2$, $y=1$, $z=\frac{1}{2}$, etc. 102. $x=1\frac{1}{2}$, $y=\frac{1}{2}$, $z=3$, etc.

103. $x=6$, $y=5$, $z=3$, etc.

104. $x=2$ or 3, $y=3$ or 2, $z=1$ or 4.

105. $x=2$, $y=3$, $z=4$, etc.

Ex. LXIX. p. 58.

1. (1) $x^2-21x+80=0$. (2) $x^2-\frac{17}{4}x+1=0$.
   (3) $x^2-\frac{31}{4}x-2=0$. (4) $x^2+9x+20=0$.
   (5) $x^2-2ax+a^2=b^2$. (6) $x^2-5=0$.
   (7) $x^2-2\sqrt{3}x+2=0$. (8) $(a^2-b^2)x^2-2(a^2+b^2)x=b^2-a^2$.
   (9) $x^2-3x+1=0$. (10) $x^2-2\sqrt{(a+b)}x+2b=0$.

2. $12p^2x^2-7px+1=0$. 3. $x^2-2(p\pm q)x\pm 4pq=0$.

4. $(q^2-q)^2x^2-(p^2-2q+2q^2)x+1=0$.

5. $\frac{1}{2}(a^2+b^2+5ab)$. 7. 1900, 35000, 1330000.

8. $qx^2-(p^2-3q)px+q^2=0$. 10. $\dfrac{2(ac+bc+ab)}{a+b+c}, \dfrac{3abc}{a+b+c}$.

11. $x^2-2(a+b)x+2ab=0$. 12. $x^2-2cx-b^2+c^2+2c=0$.

13. $p(p^4-5p^2q+5q^2)$. 14. q^2-4q+2.

15. $x^2-(3pq-p^3+p^2-2q)x+p(3q-p^2)(p^2-2q)=0$.

16. $np^2=(n+1)^2q$. 17. 32. 20. (1) $(x-35)(x-8)$.
   (2) $(3x-5)(5x-3)$. (3) $(4x+5)(2x-3)$.

(4) $(4x+3)(3x+4)$. (5) $(x+2a-3b)(x-3a+2b)$.

(6) $\{x-(a-b+c)\}\{x-(a-b-c)\}$.

(7) $(x-2a-1)(x-a+3)$. (8) $(x+a-b)(x+a-c)$.

(9) $(x+p-q)(x+a-b)$.

(10) $(2x-a+b-c)(2x-a+b+c)$.

(11) $(x+y-z+1)(x-y+z)$. (12) $(x+a-y+z)(x+2a+y-z)$.

21. $12\frac{1}{2}$. 22. 0 or 6. 23. ± 15. 24. 8 or $-\frac{4}{3}$.

25. $a^3+b^3+c^3=3abc$. 26. $a^2+b^2+c^2=ab+ac+bc$.

27. 6 or $\frac{6}{5}$. 28. $a^3+b^3+c^3=3abc$.

31. $(c^3-b^3)^2=4bc(ac-b^2)(c^2-ab)$. 33 8.

34. 4 or $4\frac{3}{4}$. 35. 0 or 6. 38. ± 2.

52. x must not lie between 4 and 1 and y between $\frac{4}{5}$ and 1.

Ex. LXX. p. 68.

1. 10 and 20. 2. 5 and 25. 3. 11 and 17 or 1 and 7.

4. ± 12. 5. ± 8 and ± 12. 6. 7 and 3. 7. 4 and 5.

8. 48. 9. 256. 10. 40 and 20. 11. 50 and 64.

12. 15. 13. 12 and 16 yds. 14. Rs 120, 60, 30.

15. Rs 550 and Rs 450. 16. 70 min. and 65 min.

17. 4 miles an hour. 18. 6 hrs. and 12 hrs.

19. 24. 20. 120 and 80.

21. 5. 22. 5 annas. 23. 3 yds. and 4 yds.

24. $4\frac{1}{2}$ A. M. and 5 A. M. 25. A, £120 ; B, £80.

26. Rs 2500 and Rs 2400. 27. 448. 28. 16.

29. 125 and 8. 30. 18. 31. 33.

32. $1\frac{2}{5}$ as. and $5\frac{1}{3}$as. 33. 8. 34. 20, 16 and 10ft

35. 720 miles. 36. 4 hrs. and 6 hrs. 37. 8 and 6 inches.

38. A's rate 7 miles and B's rate $9\frac{1}{3}$ miles.

39. 360 yds. 40. 3 miles an hour. 41. 180000.

42. 1200. 43. 4, 6 and 8. 44. 2, 3, 4, 5.

45. 28 men and 45 lbs. or 36 men and 77 lbs.

46. 385 ft. or 324 ft. 47. 3 miles ; $21\frac{9}{11}$ minutes after 2 A.M.

48. 30 miles an hour ; at half past 9 o'clock.

Ex. LXXI. p. 80.

1. $6\frac{2}{3}$. 2. 4. 3. $9x=10y$. 4. 9. 5. Rs30.
6. $bxz=acy$. 7. $x^3=\frac{64}{9}y^2$. 8. $a=\frac{1}{24}bc^2$.
10. $x=\dfrac{8a^2-b^2+ab}{b-a}$. 12. $b=\pm\frac{4}{5}a$.
19. $y^2+5y=xy+3$. 20. $y=8$; $x=6$ or -1.
21. $x=1+2ab-3ab^{-1}$. 22. $y=6x+12x^2$; $x=\frac{1}{2}$ or -1.
23. $y=3+6x+12x^2$. 24. $y=(n^2-n+1)a$.
25. $\sqrt{(R^2+r^2)}$. 26. 7 : 8. 27. 50 inches.
28. £2000, £100. 29. 143. 30. 40 per cent.

Ex. LXXII. p. 88.

1. (1) 41. (2) 97. (3) 100. (4) $-7\frac{5}{6}$.
(5) $23\frac{1}{2}$. (6) $3n-2$. (7) $4n-1$. (8) $4n$.
(9) 0. (10) $\frac{1}{3}(5n-4)$. (11) $x^2+y^2+2xy(2-n)$.
(12) $\dfrac{(2a-b)n-a}{a+b}$. 2. 1. 3. -3.
4. 100. 5. 3. 6. 4, 6, 8, 10, 12 and 14.
7. $\frac{5}{16}, \frac{1}{8}, -\frac{1}{16}, -\frac{1}{4}, -\frac{7}{16}, -\frac{5}{8}$ and $-\frac{13}{16}$.
8. $\frac{3}{5}, \frac{1}{5}, -\frac{1}{5}$ and $-\frac{3}{5}$. 9. 8 the 1st. term, 7 the com. diff.
10. 392. 11. -120. 12. 180. 13. -460.
14. 280. 15. 3775. 16. 10000. 17. 28.
18. -7. 19. $\frac{1}{6}(6n^2+n)$. 20. $\frac{1}{2}(n-1)$.
21. $\frac{1}{4}n(n-1)(n-4)$. 22. $\frac{1}{6}n(n-1)(n-4)$. 23. 8.
24. 1st. term $=1$. 25. 10. 26. 6. 27. 8.
28. 21. 29. 9. 30. 1, 3 and 5. 31. 3, 5 and 7.
32. 4, 8, 12 and 16. 33. 1, 3, 5 and 7.
34. 16 and 20 inches. 35. 6 miles 40 yds.
36. 7, 11, 15, 19 and 23. 37. 1, 3, 5, 7... 38. 13.
41. 1, 3, 5, 7... 42. 1, 3, 5, 7, 9. 48. $\frac{1}{2}n(3n-1)$.
49. m^{th} term $2p+r(2m-1)$; sum $=(2p+mr)m$.
50. Sum $=\dfrac{n}{2}\left\{\dfrac{(n+1)(P-Q)-2(Pq-Qp)}{p-q}\right\}$.
56. $-2n$. 58. $n+1$. 60. $\frac{1}{3}n(n+1)(n+2)$.
61. $\frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}$. 62. $na^2+abn(n-1)+\frac{1}{6}n(n-1)(2n-1)b^2$.

Ex. LXXIII. p. 97.

1. $\frac{1}{3^{11}}\{3^{12}-2^{12}\}$. 2. $\frac{2^{10}-1}{3\times 2^9}$. 3. $\frac{2^{12}-1}{2^{12}}$.

4. $\frac{2^{10}-1}{2^{10}}\times\frac{4}{3}$. 5. $\frac{3^n-2^n}{3^{n-3}}$. 6. $\frac{3}{4^{n-1}}\{7^n-4^n\}$.

7. $\frac{a^2}{a-b}$. 8. $166\frac{2}{3}$. 9. $\frac{8}{7}$. 10. $\frac{1}{3}$.

11. $\frac{ab}{a+b}$. 12. $\frac{2}{1-a}$. 13. $\frac{9}{10}$. 14. $255\sqrt{2}$.

15. $\frac{211}{16}(\sqrt{3}+\sqrt{2})$. 16. $-\frac{85}{24}(\sqrt{6}-2)$.

17. $\frac{1111}{5}\{\sqrt{(15)}-5\sqrt{6}\}$.

18. $\sqrt{(\frac{3}{2})}\left\{\frac{4^{\frac{n}{2}}-3^{\frac{n}{2}}}{2-\sqrt{3}}\right\}\times 3^{-\frac{n}{2}}$ 19. $\frac{\sqrt{(2^n)}-1}{\sqrt{2}-1}\times 2^{-\frac{n}{2}}$.

20. $\frac{3a\sqrt{a}}{x\sqrt{(6a)}-2x\sqrt{x}}$. 21. $3\sqrt{2}+4$. 22. $\frac{5}{8}$. 23 $\frac{2^{n+1}-2-n}{2^{n-1}}$.

24. $\frac{3^n-1-n}{3^{n-1}}$. 25. $\frac{8x-2x^2}{(2-x)^2}-\frac{x^{n+1}(4+2n-nx-x)}{2^{n-1}(2-x)^2}$.

26. $\frac{2^n\pm 1}{9\times 2^{n-2}}\pm\frac{n}{3\times 2^{n-1}}$. 27. $\frac{9x}{(3-x)^2}$.

28. 3, 6, 12, 24, 48,... 29. 4, 2, 1.

30. 1, 2, 4. 31. 2, 4, 8, 16 or $\frac{810}{13}, -\frac{540}{13}, \frac{360}{13}, -\frac{240}{13}$.

33. $\frac{1}{3}$, 1, 3, 9.

34. $\frac{2}{5}+\frac{4}{25}+\frac{8}{125}+\ldots$ or $\frac{14}{15}-\frac{28}{75}+\frac{56}{375}-\ldots$

35. $\frac{1}{10}$ or 10 per cent.

36. Common ratio $=\frac{2}{3}$ or $-\frac{1}{2}$.

37. Rs 256, Rs 192, Rs 144, Rs 108. 40. 139.

41. $\frac{ap}{2}(1+p)+\frac{a^{p+1}-a}{a-1}$. 42. $\frac{7}{2}$ or $-\frac{1}{2}$.

44. $\frac{Ss}{S+s-1}$. 47. $\frac{a^{4n+2}-1}{a^{2n}(a^2-1)}-2n-1$.

48. $\frac{ac}{1-r}+\frac{bcr}{(1-r)^2}$. 52. $\frac{1}{2}n(n+1)\frac{r^n-1}{r-1}a$.

Ex. LXXIV. p. 102.

1. $1\frac{3}{5}$, $1\frac{1}{3}$. 2. $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, &c. 3. $-\frac{15}{2}$.

4. 6. 5. 6 or $\frac{3}{2}$. 6. 4 and 12.

7. 2, 3 and 6. 8. 2 and 18. 9. 2, 3, 6.

10. 10 and 40. 13. $\dfrac{NR(n-r)}{Rn-Nr}$. 16. 4, 6 and 12.

Ex. LXXV. p. 103.

2. (1) 4 or $\sqrt[3]{(49)}$. (2) $x = \pm 2, y = \pm 4, z = \pm 6$.

3. $(a-b-c)(a-b-c)$. 4. $e^{3x} - pe^{2x} + qe^{x} = r$.

5. (1) 4 or $\frac{1}{4}$. (2) $\pm \frac{1}{2} a\sqrt{3}$. (3) 2, 3, 5.

6. $\dfrac{a - r^n(a-nb)}{1-r} - \dfrac{br(1-r^n)}{(1-r)^2}$. 8. (1) $-\frac{1}{2}$ or $-\frac{1}{6}$.

(2) $x = \pm 2, y = \pm 1$. (3) $x = \mp \frac{1}{2}$ or $\pm \frac{19}{2}, y = \pm 2$ or ± 7.

9. Skilled workmen 14, labourers 10. 10. $16\frac{1}{5}$.

11. (1) 1 or 3. (2) $x = \pm \dfrac{a \pm b}{ab}, y = \mp \dfrac{a \pm b}{ab}$.

12. $a^{\frac{5}{2}} - a^2 b^{\frac{1}{3}} + a^{\frac{3}{2}} b^{\frac{2}{3}} - ab + a^{\frac{1}{2}} b^{\frac{4}{3}} - b^{\frac{5}{3}}$.

13. $\sqrt{3} - 1 + \sqrt{5}$ or $1 - \sqrt{3} - \sqrt{5}$.

15. (1) ± 1 or $1 - \sqrt{3}$. (2) $\frac{1}{2}$ or -1. (3) -1 or 2.

16. $101\left(\dfrac{101^{10} - 100^{10}}{100^{10}}\right)$. 17. $a^2 + b^2 + c^2 - abc = 4$.

19. (1) 1 or $\pm \sqrt{(-1)}$. (2) 1 or -2.

20. 59, 72 and 85. 21. $\dfrac{3a^2 - b^2}{2a}$.

22. $\dfrac{a - ar^n - 2bnr^n}{1-r} + 2br\dfrac{1-r^n}{(1-r)^2}$. 23. 1.

25. $x^3 - px^2 + qx - r = 0$. 28. b.

30. $2(ac - bd)$. 31. $x = \dfrac{c^2 - b^2}{2(a-c)}$; any thing.

Ex. LXXVI. p. 114.

1. $\lfloor\underline{8}$ or 40320. 2. 360 ; 5040 ; 7560 ; 9979200.
3. 7. 4. 5. 5. 720 6. 5040. 7. 5.
8. 216. 9. 420 ; 15. 10. $120+15=135$.
11. 210. 12. 5. 13. 31×16 ; 31×15.
14. 20736. 15. 14. 16. 144000.
17. $6+15+20+15+6+1=63$. 18. 30240 ; 3024.
19. $n=2m$. 20. 120. 21. 907200.
22. $n=9$. 23. 1920. 24. $n=10$.
25. $m=4$, $n=6$. 27. $\frac{\lfloor\underline{20}}{\lfloor\underline{9}\,\lfloor\underline{11}}$, $\frac{\lfloor\underline{19}}{\lfloor\underline{9}\,\lfloor\underline{10}}$, $\frac{\lfloor\underline{17}}{\lfloor\underline{9}\,\lfloor\underline{8}}$. 28. 980.
29. 10 together ; 43758. 30. $\frac{1}{6}m(m-1)(m-2)$; $\frac{1}{2}(m^2-3m)$.
31. 4845 ; 969.
32. 969; 3876. 33. $\frac{\lfloor\underline{64}}{\lfloor\underline{8}\,\lfloor\underline{56}}$; $\frac{\lfloor\underline{61}}{\lfloor\underline{5}\,\lfloor\underline{56}}$. 34. 4320.
35. 64. 36. 256. 37. 360.
38. 140. 39. 6720. 40. $\frac{\lfloor\underline{20}}{12\times8}$.
41. $\frac{\lfloor\underline{26}}{(\lfloor\underline{13})^2}$; $\frac{\lfloor\underline{18}}{\lfloor\underline{5}\,\lfloor\underline{13}}$. 42. 30. 43. $74\lfloor\underline{8}$.

Ex. LXXVI. (B) p. 128.

1. $1+20x+180x^2+960x^3+\dots\dots+2^{10}x^{10}$.
2. $600,000,000x^8$. 3. $\frac{\lfloor\underline{30}}{(\lfloor\underline{15})^2}x^{15}y^{15}$. 4. $\frac{100\times99}{1\times2}x^{-90}$.
5. $\frac{\lfloor\underline{32}}{\lfloor\underline{14}\,\lfloor\underline{18}}a^{36}b^{28}$. 6. $\frac{\lfloor\underline{17}}{\lfloor\underline{8}\,\lfloor\underline{9}}x$ and $-\frac{\lfloor\underline{17}}{\lfloor\underline{9}\,\lfloor\underline{8}}x^{-1}$.
7. $\frac{\lfloor\underline{24}}{(\lfloor\underline{12})^2}x^{-12}$, the 13^{nth} term is the only middle term.
8. $\frac{1}{6}(2n+1)n(2n-1)(n-1)a^{4n-6}x^8$.
9. $\frac{\lfloor\underline{2n}}{\lfloor\underline{r}\,\lfloor\underline{2n-r}}a^{4n-4r}(-1)^r$. 10. $\frac{\lfloor\underline{20}}{\lfloor\underline{5}\,\lfloor\underline{15}}$.
11. There is no term involving x^{11}, for only the even powers of x are involved.

12. $16a^2(4a^4-3x^4)$. 13. $32a^2x^2(10a^4x^4-3a^8-3x^8)$.

14. -256. 15. Product $=(a^4+x^4)^{10}$; it does not involve x^{10}. 16. -38760.

17. $2\left\{a^n-\dfrac{n(n-1)}{1\cdot2}a^{n-2}b^2+\dfrac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}a^{n-4}b^4-\right\}$.

18. $2\left\{na^{n-1}b-\dfrac{n(n-1)(n-2)}{1\cdot2\cdot3}a^{n-3}b^3\right.$
$\left.+\dfrac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot2\cdot3\cdot4\cdot5}a^{n-5}b^5-\ldots\right\}$.

19. $\dfrac{\lfloor 4n+2}{(\lfloor 2n+1)^2}$ 22. $(1+3)^6$.

Ex. LXXVII. p. 140.

1. $1+\frac{1}{4}x-\frac{3}{32}x^2+\frac{7}{128}x^3$. 2. $1-\frac{2}{3}x-\frac{1}{9}x^2-\frac{4}{81}x^3$.

3. $1+2x+3x^2+4x^3$.

4. $1-nx+\frac{1}{2}n(n+1)x^2-\frac{1}{6}n(n+1)(n+2)x^3$.

5. $1+\frac{3}{4}x+\frac{21}{32}x^2+\frac{77}{128}x^3$. 6. $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3$.

7. $2a^{\frac{1}{3}}-\frac{1}{4}a^{-\frac{2}{3}}x-\frac{1}{32}a^{-\frac{5}{3}}x^2-\frac{5}{768}a^{-\frac{8}{3}}x^3$.

8. $a^{-\frac{3}{2}}+\frac{3}{4}a^{-\frac{7}{2}}xy+\frac{21}{32}a^{-\frac{11}{2}}x^2y^2+\frac{77}{128}a^{-\frac{15}{2}}x^3y^3$.

9. $1+5nx^2+25\dfrac{n(n+1)}{1\cdot2}x^4+\dfrac{n(n+1)(n+2)}{1\cdot2\cdot3}125x^6$.

10. $(2a)^{\frac{3}{4}}\left\{1-\dfrac{9x}{8a}-\dfrac{27x^2}{128a^2}-\dfrac{135x^3}{1024a^3}\right\}$.

11. $a^{-\frac{1}{2}}\left\{1+\dfrac{b^2y}{4a^2}+\dfrac{5b^4y^2}{32a^4}+\dfrac{15b^6y^3}{128a^6}\right\}$.

12. $1-\frac{30}{4}x+\frac{1053}{32}x^2-\frac{5265}{128}x^3$. 13. $\frac{1}{2}(r+1)(r+2)x^r(-1)^r$.

14. $\frac{1}{6}(r+1)(r+2)(r+3)x^r$.

15. $\dfrac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{\lfloor r}x^r$.

16. $\dfrac{2\times5\times\ldots\ldots(3r-1)}{3^r\lfloor r}a^rx^{3r}$.

17. $-\dfrac{(n-1)(2n-1)(3n-1)\cdots(rn-n-1)}{n^r\lfloor r}x^{2r}$.

18. $(-1)^r\times\dfrac{1\times4\times7\cdots(3r-2)}{3^r\lfloor r}x^r$.

19. $\dfrac{1\times4\times7\times\ldots\ldots(3r-2)}{3^r|\underline{r}}x^{r+2}$.

20. $\dfrac{2\times5\times8\times\ldots\ldots(3r-1)}{3^r|\underline{r}}a^{\frac{20}{3}}x^{5r}$.

21. $\dfrac{1\times4\times7\times\ldots\ldots(3r-2)2^r}{3^r|\underline{r}}a^{-\frac{1}{3}-r}b^r x^{\frac{3}{3}}$.

22. $\frac{1}{3}(r+1)(2r+1)(2r+3)x^r$.

23. $\dfrac{1\cdot1\cdot2\cdot5\cdot8\cdot11\cdot14\cdot17\cdot20\cdot23\cdot26}{|\underline{10}}$. 24. $-5120a^{-22}x^{16}$.

25. 2·0361. 26. 9·9966. 27. 10·000159.

28. 2·0066. 30. $A=\frac{3}{2}$, $B=\frac{15}{8}$, $C=\frac{35}{16}$.

31. $-\dfrac{1\cdot3\cdot5\ldots\ldots15\times\sqrt{2}}{2^{18}|\underline{9}}a^{-\frac{17}{2}}x^{\frac{19}{2}}$. 32. $2n^2+2n+1$.

33. $\dfrac{1\cdot3\cdot5\ldots\ldots(2n-1)}{2^n|\underline{n}}$. 34. -8. 35. $4n+1$.

36. 455. 37. $(n+1)^2$; there is only one term involving x^n.

38. $a^{-r}\times\dfrac{3\times5\times7\ldots(r-1)}{2^{\frac{1}{2}r-1}|\underline{\tfrac{1}{2}r-1}}\left\{\dfrac{4r+1}{r}\right\}$ when r is even, or

$a^{-r}\left\{\dfrac{3\cdot5\cdot7\ldots\ldots\ldots(r-2)}{2^{|\underline{\frac{1}{2}(r-3)}}|\underline{\tfrac{1}{2}(r-3)}}\right\}\left\{\dfrac{4r-1}{r-1}\right\}$ when r is odd. 39. 1*st*.

40. 1*st*. term. 41. 2*nd*. term. 42. $-\frac{2431}{8192}$. 43. 7*th*.

44. The coefficient of the $(n+2)^{th}$ term

$$=\frac{1\cdot3\cdot5\ldots\ldots(2n+1)}{|\underline{n+1}}(-1)^{n+1}.$$

46. $\frac{1}{6}(n+1)(n+2)(n+3)$.

Ex. LXXVIII. p. 148.

1. 10. 2. 4. 3. $\log 5=\cdot6989700$; $\log 16=1\cdot2041200$; $\log 32=1\cdot5051500$.

4. $\log 12=1\cdot0791812$; $\log 15=1\cdot1760913$; $\log 75=1\cdot8750613$. $\log 1\cdot8=\cdot2552726$. 5. 3. 6. 2.

7. 3; 4; 2; -2. 8. 3·3344538, $\bar{1}\cdot57501226$.

9. ·3010300, ·4771213. 10. ·4771213; ·6989700.

11. ·2761407. 12. $\bar{3}\cdot8890756$. $\bar{1}\cdot7168662$.

13. 112. 14. 1·0749066; ·8969100. 15. $x=3$.

16. 2·9 nearly. ·3010300. 18. $4\frac{1}{3}$. 19. 2·48595.
20. 259·569. 21. 4·37437. 22. 2·57 nearly.
23. 5444525. 24. 1·58. 25. 10·5675. 26. 166 years.

Ex. LXXIX. p. 156.

1. 1·00100050. 2. ·9048367. 3. Less than ·000028.

4. $< \dfrac{1}{11 \times 2^5 \lfloor 5}$. 5. $< \dfrac{1}{10^7 \times 79 \times \lfloor 7}$. 6. 5 terms.

9. $\dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 4} + \dfrac{1}{\lfloor 6} + \&c.$ 12. $n=2, 4$ or 8, $x=10, 8$ or 4.

13. 1·386294. 14. 1·098612. 15. 3·218875.
16. 2·484906. 17. 3·583518. 20. ·002.

Ex. LXXX. p. 161.

1. Rs 4500. 2. Rs 15625. 3. Rs 135.
4. Rs 630 8 as. 5. Rs 4936 4 as. 7 p.
6. Rs 2631 13 as. 9 p. 7. 4 per cent. 8. 8 years.
9. 14·2 years; 22·5 years 10. Rs 265329 12 as.
11. 100 years. 12. 8 years.

Ex. LXXXI. p. 168.

1. Rs 12480. 2. Rs $71111\frac{1}{9}$. 3. Rs 56371.
4. 18·05 years. 5. 14·2 years. 6. Rs 100000.
7. Rs 2000. 8. Rs 8110·9. 9. Rs 6697 2 as. nearly.
10. Rs 12784 nearly; the interest of the last investment is to be taken into account.
11. Rs 16470; the interest is calculated half-yearly.
12. Rs 29272. 13. $23\frac{1}{2}$ years.
14. 75921; $23\frac{1}{4}$ years. 15. Rs 614 nearly.

Ex. LXXXII. p. 173.

1. (1) $x=10, 7, 4$ or 1 and $y=1, 3, 5$ or 7.
(2) $x=4, y=3$. (3) $x=4, y=2$. (4) $x=4, y=3$.
(5) $x=4, 7$, &c.; $y=2, 9$, &c. (6) $x=1, 6$, &c.; $y=1, 10$, &c.
(7) $x=4, 15$, &c.; $y=3, 12$, &c. (8) $x=3, 16$, &c.; $y=2, 13$, &c.
(9) $x=3, 16$, &c.; $y=1, 15$, &c. (10) $x=1, 8$, &c.; $y=2, 52$, &c.
(11) $x=5, 15$, &c.; $y=1, 22$, &c. (12) $x=9, 18$, &c.; $y=59, 132$, etc.

2. Three. 3. Five. 4. One.
5. To pay 10 G. M. and receive 11 notes. 6. 19, 39, &c.
7. 13. 8. Number of men 3 or 8, that of boys 18 or 4.
9. 85 or 145. 10. Three. 11. 5, 7, 4.
12. 2, 14, 20 ; or 4, 7, 25. 13. 3 or 6 horses and 6 or 1 bullocks respectively. 14. 9 and 3. 15. 79.
16. 45, 55. 17 Two. 18. 344.
19. 5, 2, 1 ; 1, 5, 1 ; 6, 1, 2 ; 2, 4, 2. 20. 3, 12, 5 ; or 6, 4, 10.

MISCELLANEOUS EXAMPLES. p. 175.

1. (1) $x=4$ or $-\frac{25}{7}$. (2) $x=3, -\frac{11}{2}, \frac{1}{4}\{-5 \pm \sqrt{(33)}\}$.
(3) $x=\pm 2$ or $\pm\frac{10}{3}$; $y=\pm 3$; or $\pm\frac{7}{5}$.
2. 126 officers and 4000 men. 3. 1, 3, 5, 7.
4. (1) $x=\left(\frac{a^2-b^2}{am-bn}\right)^2$; $y=\pm\sqrt{\left(\frac{a^2-b^2}{an-bm}\right)}$.
(2) $x=2$ or $-3\frac{1}{15}$. (3) $x=\frac{ma}{n}$ or $\frac{mn}{a}$. 5. 1880.
6. $\sqrt{10}-\sqrt{2}$. (1) $\frac{1}{8}(9+\sqrt{3})$. 7. 1 or -3.
8. (1) $x=\frac{1}{4}$ or $\frac{1}{9}$. (2) $x=2$ or 16, $y=2$ or $\frac{1}{2}$.
(3) $x=3$ or 4, $y=4$ or 3.
10. (1) $\frac{1}{6}(5a+b)$; $\frac{1}{3}(2a+b)$, $\frac{1}{2}(a+b)$, $\frac{1}{3}(2b+a)$, $\frac{1}{6}(5b+a)$.
(2) 90. (3) $9\frac{43}{48}$. (4) $\frac{3}{2}\{3\sqrt{2}+\sqrt{6}\}$.
11. $\frac{m^2-n^2}{m}$. 12. $\frac{\lfloor 50}{\lfloor 6 \lfloor 44}$, $\frac{\lfloor 49}{\lfloor 5 \lfloor 44}$.
13. (1) $\frac{1}{4}\{3\pm\sqrt{(13)}\}$. (2) $x=\pm 5$ or $\pm\sqrt{2}$, $y=\pm 3$ or $\mp 4\sqrt{2}$.
14. $\pm 1\frac{2}{3}$ and ± 3. 15. (1) 240. (2) 2. (3) $\frac{5}{6}$.
16. (1) 5 or $-3\frac{5}{8}$. (2) $4, -9\frac{1}{2}$ or $\frac{1}{4}\{-11\pm\sqrt{(89)}\}$.
(3) $x=\pm 1$ or ± 4, $y=\pm 3$ or ± 2.
17. ± 3 and $\pm 2\frac{1}{3}$. 18. $\frac{7}{9}$. 19. $\frac{\lfloor 11}{\lfloor 2}$. 20. $\frac{\lfloor 40}{\lfloor 33 \lfloor 7}$; $\frac{\lfloor 39}{\lfloor 33 \lfloor 6}$.
21. $\frac{5}{2}$; log 128 = 2·1072100 ; log 125 = 2·0969100 ;
log 256 = 2·4082400.
22. Rs 1479 7as. 8p. 23. (1) $x=a+b$ or $\frac{2ab}{a+b}$.
(2) $x=4$ or $\frac{2}{3}$; $y=1$ or $-2\frac{2}{3}$. (3) $x=36$, $y=4$.

24. 25 gals. rum; 5 gals. brandy. 26. 10 and 6 or $-\frac{5}{2}$ and $\frac{3}{2}$.

27. $x^2-2ax \mp a^2-b=0$.

28. $(3a)^{\frac{4}{5}}\left\{1-\frac{16x}{15a}-\frac{32x^2}{225a^2}-\frac{4}{125}\times\left(\frac{4x}{3a}\right)^3-\dots\right\}$.

$\frac{1}{2}(r+1)(r+2)x^r$. 29. 1·9285266 ; $\overline{1}$·9830103.

30. (1) 4 or $-\frac{2}{11}$. (2) $a+b$ or $\frac{a^2+b^2}{a+b}$.

(3) $x=3$ or -2, $y=1$ or 6. 31. 4 and 5 yds.

32. (1) 0. (2) $\frac{2}{3}(2+11\sqrt{3})$. (3) 25.

33. 8. 35. $x=\frac{\log 2}{\log 3}$, $y=\frac{\log 3}{\log 2}$. 36. $n=14$

37. $1+7x+28x^2+84x^3$; $\frac{1435}{128}$. 38. $\frac{|\underline{16}}{|\underline{6}\,|\underline{10}}$

39. 2·3856063. 40. Rs 32000. 41. (1) 6 or $\frac{37}{11}$.

(2) $x=\pm 3$ or $\pm 9\sqrt{2}$, $y=\pm 2$ or $\pm\frac{7}{2}\sqrt{(-2)}$.

(3) $x=25$ or $\frac{25}{9}$, $y=1$ or $\frac{100}{9}$. 42. 64 and 50.

43. $x^2-x-5=3\sqrt{3}$. 44. $\frac{(n+1)^2}{2(n-1)^2}$.

45. p^r. 46. $b=7\frac{1}{2}$.

47. $\frac{|\underline{8}}{(|\underline{2})^2}$. 48. $1+\frac{3}{2}x+\frac{15}{8}x^2+\frac{35}{16}x^3$.

49. (1) $x=2·40824$; (2) $x=3·773$.

50. The present value = Rs 100 to be paid now together with the present value of an annuity of Rs 100 for 9 years = Rs 810·782.

51. (1) $x=\frac{a^2}{b}$ or $-\frac{b^2}{a}$. (2) $x=\pm 1, y=\pm 2$.

(3) $x=\pm\sqrt{\{\frac{1}{2}(1\pm\sqrt{5})\}}$.

(4) $x=\pm 3$, $\pm\sqrt{6}$, $\pm\sqrt{\{\frac{1}{2}(15\pm 3\sqrt{5})\}}$ or $\pm\sqrt{\{\frac{1}{2}[-13\pm \sqrt{(-47)}\}}$; $y=2$, -1, $\frac{1}{2}(1\pm 3\sqrt{5})$ or $\frac{1}{2}\{1\pm\sqrt{(-47)}$.

52. 18 ; Rs 40. 53. $12x^2-25x+12=0$.

55. $n^2=(c-a)^2$ or $a^2+b^2+c^2=ab+bc+ac$. 57. 20.

58. 1·9743. 59. 9·15054. 60. Rs 4802 6as.

61. (1) $x=\frac{1}{2}\left\{a+b\pm\sqrt{\left((a+b)^2-\frac{8a^2b^2}{a^2+b^2}\right)}\right\}$.

(2) $x=6, -7$ or $-\frac{1}{2}\{1\pm\sqrt{(221)}\}$.
(3) $x=2, -10, 16, -24$ &c.; $y=1, 25, 64, 144$ &c.

62. $x^2-2(a^2+b^2)x+(a^2-b^2)^2=0$.

63. $x^2-(\frac{1}{2}p+\sqrt{q})x+\frac{1}{2}p\sqrt{q}=0$. 64. $x=a, 2a$ &c.

65. 8. 66. $\dfrac{a^3}{a^2+b^2}$. 67. 1·0002.

68. $1+x+x^2+\frac{1}{2}x^3$. 69. $\dfrac{\lfloor 98}{\lfloor 10\lfloor 88}, \dfrac{\lfloor 98}{\lfloor 12\lfloor 86}$. 70. 11 years.

71. (1) $x=0, a, \frac{1}{2}\{1\pm\sqrt{(33)}\}$. (2) $x=1$ or $3, y=3$ or 1.
(3) $\sqrt{(xy)}=6$ or $-7, x=4\frac{1}{4}$ &c.

72. $a^8b^8+a^3c^3+b^3c^3=3a^2b^2c^2$. 74. $a^2+b^2+c^2\pm 2abc=1$.

75. Rs $4\frac{1}{2}$ price of goat and Rs $7\frac{1}{2}$ price of sheep.

76. (1) $x=2, -5$ or $\frac{3}{2}\{-1\pm\sqrt{(241)}\}$.
(2) $x=2-\sqrt{3}$ or $2\sqrt{3}-4$. (3) $x=1$ or -2.

78. 3. 79. $(a+b)^3+(b+c)^3+(a+c)^3=3(a+b)(b+c)(a+c)$.

80. $n=8$. 81. First $=15(\sqrt{2}-1)$, common ratio $=\sqrt{2}$.

82. 1536·58. 84. ·0999996000.

85. 11 years. 86. (1) $x=\frac{3}{2}, -1$ or $-\frac{1}{2}$.
(2) $x=\frac{1}{2}a, y=\frac{1}{2}(a-b)^2$. (3) $z=\frac{1}{2}(n^2-3n)$, etc.

87. $a^3+b^3+c^3=3abc$. 88. $a^3+b^3+c^3=3abc$.

89. $a^3+b^3+c^3=(a+b)(b+c)(a+c)$.

91. $s=\dfrac{a-ar^n+nbr^n}{1-r}-\dfrac{br(1-r^n)}{(1-r)^2}$. 92. (1) $x=4$ or $\sqrt[3]{(49)}$.
(2) $x=\pm 2, y=\pm 4, z=\pm 6$.
(3) $x=1, 4, \frac{4}{9}, \frac{4}{25}, \frac{4}{49}$; $y=-1, 2, -\frac{2}{3}, \frac{2}{5}$ or $-\frac{2}{7}$.

93. Rs 6000 stock of A, Rs 4000 stock of B; A's gain Rs 600
B's Rs 400. 94. 36, 24 and 16 gallons.

95. Rs 643 15 as. nearly. 96. $x=1$ or $5, y=3, z=5$ or 1.

97. -20. 99. 1 year 221 days. 100. Rs 4451 12 as.

F. A. PAPERS. p. 185.

1861. 1. $3\sqrt{3}-1$; $1+\sqrt{(-1)}$.

2. $\dfrac{1}{\sqrt{2}}$; a^n+1+a^{-n}; $x^{\frac{2}{3}}-a^{\frac{1}{3}}x^{\frac{1}{3}}+a^{\frac{2}{3}}$. 3. (1) $x=\frac{2}{3}$ or $\frac{2}{15}$.
(2) $x=0, \pm c\sqrt{\{2\pm\sqrt{(-1)}\}}$; $y=0, \pm\frac{1}{2}c\sqrt{\{10\pm 5\sqrt{(-1)}\}}$.

4. $\frac{ra(r^n-1)}{(r-1)^2}-\frac{na}{r-1}$. 5. $\frac{20\times19}{1\times2}$; $\frac{19\times18}{1\times2}$.

6. $\frac{11\times8\times\ldots(14-3r)}{\sqrt{(3^r)}\,\lfloor r}x^{\frac{22+3r}{2}}y^{\frac{22-3r}{2}}$, the 2nd and the 4th terms being negative.

1862. 1. ·256. 2. $\frac{3x-5}{x^2-3x+2}$; $\frac{2x^2+3ax+7a^2}{x^2-6ax+2a^2}$.

3. $x^{\frac{mn}{m+n}}+y^{\frac{mn}{m+n}}+z^{\frac{mn}{m+n}}=p^{\frac{mn}{m+n}}$.

4. $x=\frac{1}{(b-a)^2}\{4ab\pm(a+b)\sqrt{(6ab-a^2-b^2)}\}$.

5. $x=-1, \frac{1}{2}, -\frac{3}{2}$; $y=-1, -\frac{3}{2}, -2$.

6. $\frac{17\times14\times11\times\ldots(20-3r)}{\lfloor r}x^{\frac{17-3r}{3}}y^{\frac{34-3r}{6}}z^{\frac{r}{2}}$, the 2nd, 4th and 6th terms being negative.

1863. 1. (a) $\frac{x-b}{x+b}$; $\frac{(e^x-1)(x^2-x+1)}{(e^x+1)(x-1)}$.

2. (a) $x=28$. (b) $x=9$ or 1, $y=1$ or 9.

(c) $x=4, 9$ or $\frac{1}{2}\{-1\pm\sqrt{(-67)}\}$. 3. (a) $\frac{210}{a}$.

(b) $\frac{1}{3}\{4-2\sqrt{6}\}\{-(\frac{3}{2})^{\frac{1}{2}n}-1\}$.

4. $15625-3125x+\frac{3125}{12}x^2-\frac{625}{54}x^3+\frac{125}{432}x^4$. 5. 840; 5040.

1864. 1. $1+\frac{1}{5}x-\frac{2}{25}x^2+\frac{6}{125}x^3-\frac{21}{625}x^4$. 2. $\frac{233}{1056}$.

3. 400. 4. $(AC'-A'C)^2-(B'C-BC')(A'B-AB')=0$

5. £330·34, £369·66. 6. $2046\frac{26}{460}$.

7. The circumference of the fore-wheel $=$

$\frac{40b-b^2\pm b\sqrt{(b^2+2400)}}{2(b+10)}$, when $b=\frac{5}{16}\times3{\cdot}14159$.

1865. 1. (a) $xy=6$ or 3 etc.

(b) $x=4, -2, \frac{1}{22}\{11+\sqrt{(-231)}\}$, etc.

2. £600 to A, £888 17s. $9\frac{1}{3}d.$ to B, and £1111 2s. $2\frac{2}{3}d.$ to C.

3. $m^{\frac{2}{3}}-n^{\frac{2}{3}}=4$. 4. $\frac{a-a(-r)^n}{1+r}$.

5. $3+a^{-1}x^2+\frac{2}{3}a^{-4}x^4+\frac{14}{27}a^{-7}x^6+\frac{35}{81}a^{-10}x^8+\ldots\ldots\ldots$

6. $\dfrac{1\times4\times7\times(3r-2)}{\lfloor r\times3^{r-1}}\left(\dfrac{x^2}{a^3}\right)^r$.

1866. 1. (a) $x=1$ or 5. (b) $y=0, x=2$ or -1, $z=-1$ or 2.

2. 20 in the first and 59 in the second division.

3. $\dfrac{3^n-1}{2}$. 4. 720 ; 19600.

5. $\dfrac{15\times17}{2\times2^2}a^{-\frac{99}{5}}x^{-10}$; $\dfrac{15\times17\times19\times21\times23}{2^5\lfloor 5}a^{-\frac{27}{2}}x^{-\frac{5}{2}}$;

$\dfrac{15\times17\ldots\ldots(13+r)}{2^{r-1}\times\lfloor r-1}a^{-\frac{67+8r}{5}}x^{-\frac{35-5r}{2}}$.

1867. 1. $\frac{2}{3}$ or $-\frac{7}{5}$. (1) $x=0, \pm a\sqrt{5}$.

(2) $x=\pm\sqrt{\left(\dfrac{a^2+b^2}{2}\right)}, y=\pm\dfrac{a^2-b^2}{\sqrt{\{2(a^2+b^2)\}}}$.

2. $\{a_2(b_1c_3-b_3c_1)+a_3(b_2c_1-b_1c_2)+a_1(b_3c_2-b_2c_3)\}z$
$=a_3(b_2d_1-b_1d_2)+a_2(b_1d_3-d_1b_3)+a_1(d_2b_3-b_2d_3)$.
$a_2(b_1c_3-b_3c_1)+a_3(b_2c_1-b_1c_2)+a_1(b_3c_3-b_2c_3)=0$.

3. 631. 4. $\frac{10}{5}$, $\frac{5}{3}$ and $\frac{10}{7}$ are the harm. means.

5. $\frac{1}{2}(n^2-3n)$. 6. $1-3x+3x^2+2x^3-9x^4$. 7. 1·0099

1868. 1. $x=\frac{11}{7}$ or $-\frac{7}{11}$. $x=\frac{1}{3}a\{5\pm\sqrt{(35)}\}$, &c.

2. 432. 4. $100a+1023b$. 5. 7560. 6. $\frac{55}{9}a^4(\frac{1}{2}b)^8$.

1869. 1. Rs 46 10*as.* 8*p.* 2. (1) $x=6$.

(2) $x=\pm2\sqrt{2}$ or ±3. $y=\pm\sqrt{2}$ or ±1.

8. In 3 hours, at a distance of 9 miles. 10. 90.

1870. 1. $x=\dfrac{b^2+c^2-a^2}{2bc}$, $y=\dfrac{a^2+c^2-b^2}{2ac}$ and $z=\dfrac{a^2+b^2-c^2}{2ab}$.

2. $2ax^2\sqrt{a}+(b\sqrt{a}-2a\sqrt{c})x-b\sqrt{c}=0$. 3. 12 and 3.

4. 78, 66. 5. $760xy^6$. 6. 9 years.

1871. 1. $-\dfrac{304}{625}$. 2. 1·01059. 4. $z=\dfrac{2c(a+b+c)}{a+b-c}$, &c.

5. $AB=3$ miles, $AC=4$ miles and $BC=5$ miles.

1872. 1. (1) $x=\frac{-8}{25}a$. (2) $x=0$. 3. 16. 5. 4·89898.

1873. 1. $\dfrac{(x^4-y^4)(x^4-x^2y^2-y^4)y}{4x^5(y^4+x^2y^2+x^4)}$; 9 $(l+m+n)$.

2. (1) $x = \pm 3, y = \pm 1, z = \pm 2$. (2) $x = 1$ or $\frac{1202}{625}$.
(3) $x = 1, 2$ or $\frac{1}{2}\{3 \pm \sqrt{(-1)}\}$. Sum $= 5$, product $= 5$.

3. $\frac{114}{13}, \frac{100}{13}$. 4. 720, 120.

5. $$(3ax)^{14} - 14(3ax)^{13}\left(\frac{2b^3}{x}\right) + \frac{14 \times 13}{1 \cdot 2}(3ax)^{12}\left(\frac{2b^3}{x}\right)^2$$
$$+ \frac{14 \times 13 \times 12}{1 \cdot 2 \cdot 3}(3ax)^{11}\left(\frac{2b^3}{x}\right)^3; \; -\frac{1 \times 9 \times 19 \times \cdots 89}{(10)^{10}\lfloor\underline{10}}x^{10}; \; 1 \cdot 268.$$

1874. 1. 25 shillings, 10 half-crowns, 50 six-pences and 75 four-pences.

4. (α) $x = \pm\frac{1}{\sqrt{a}}$ or $-a \pm \sqrt{\left(\frac{1}{a} + a^2\right)}$.

(β) $y = \frac{2abc}{bc + ab - ac}, \; x = \frac{2abc}{ab + ac - bc}, \; z = \frac{2abc}{bc + ac - ab}$.

5. 45 and 5. 6. $\frac{1}{2}\{(n+1)(n+2)ac^n + n(n-1)bc^{n-2}\}$.

1875. 1. (1) $\frac{a(b+c)}{c(a^2 - b^2)}z + \frac{(c+a)b}{(a^2 - b^2)c}z + \frac{z}{a-b}$
$$= \frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}, \text{ \&c.}$$

(2) $x = 5$. (3) $x = 0, 1\frac{1}{2}$; $y = 0, \frac{1}{2}$.

2. $x = \pm 1$; $x = 1, 2, 3, 4$.

4. *cabd, cadb, cbad, cbda, cdab, cdba*; 60; 36. 5. $1 \cdot 99776$.

1876. 2. $5x^2 - 1$.

3. $x^2 - 2x - 4 = 0$, if $1 - \sqrt{5}$ be the other root.

4. (1) $x = \frac{ab(c+d) - cd(a+b)}{ab - cd}$ (2) $x = 3$ or 2 &c, $y = 2$ or 3 &c.

(3) $x = \frac{12}{5}, y = \frac{12}{7}, z = -12$.

5. $2\frac{1}{4}, 2^{n+1} - 2 - n$. 6. $\frac{\lfloor\underline{14}}{\lfloor\underline{2}\lfloor\underline{2}\lfloor\underline{3}}; \; \frac{\lfloor\underline{12}}{(\lfloor\underline{2})^2}$.

7. $\frac{\lfloor\underline{n}}{\lfloor\underline{m}\lfloor\underline{n-m}}$. There is no term in the expansion with x^{17}; the coefficient of $x^{18} = 210a^{16}$.

$2^{-\frac{4}{5}}(1 + \frac{6}{5}x^2 + \frac{81}{50}x^4 + \frac{567}{250}x^6 + \ldots\ldots)$.

1877. 1. 0. 2. (1) $x = 4$ or -1.

(2) $x = \pm 4$ or $\pm\frac{38}{3}\sqrt{\frac{3}{26}}$; $y = \pm 1$ or $\pm\sqrt{\frac{3}{26}}$.

(3) $x = 4, -6$ or $-1 \pm 4\sqrt{2}$.

4. Sum $=\frac{1}{6}n(n+1)(2n+1)+n$. 5. $x=abc^{-1}$.

6. 56 ; 21. 8. 58·71.

1878. 1. ·41. 2. $(6x^4-7x^2+2)(x+3)$.

3. (1) $x=3$. (2) $x=\pm 2$ or $\pm\sqrt{\frac{125}{68}}$; $y=\pm 3$ or $\mp\sqrt{\frac{17}{5}}$.

(3) $x=-1$ or 3, $y=3$ or -1 and $z=1$; $x=1$ or 3, $y=3$ or 1, $z=-1$; also $x+y=0$, &c.

4. $x^2+4ax+2b=0$. 6. 2 ; 4 ; (1) $4\{1-(-\frac{1}{3})^n\}$; 4.

(2) $\frac{1}{6}n(2n^2+9n+1)$. 8. $-945a^{-24}b^3$. $-30618a^{-30}b^5$.

$$\frac{3\times 1\times 1\times 3\times \ldots\ldots(2r-7)}{2^{r-1}\lfloor r-1}x^{r-1}.$$

1879. 1. 5 per cent. 2. ·267. 3. (1) $x=6$.

(2) $x=\frac{1}{2}$ or $-\frac{1}{3}$, $y=\frac{1}{3}$ or $-\frac{1}{2}$. (3) $x=5$, $y=9$, $z=11$.

5. (1) $\frac{1}{3}n(4n^2-1)$. (2) $\frac{122}{7}$. 6. 1638 ; 39.

7. $$\frac{n(n-1)\ldots\ldots(n-r+2)}{\lfloor r-1}(4x)^{n-r+1}(-3y)^{r-1};$$

$$-\frac{3\times 4\times\ldots\ldots(7r-17)}{1\cdot 2\cdot 3\cdot 4\cdots(r-1)}\left(\frac{3x^2}{7}\right)^{r-1};$$ the coefficient of x^{11} is 1 and there is no term involving x^{12}.

1880. 1. 1·387. 3. (1) $x=0$ or $\pm\sqrt{(ab)}$.

(2) $x=a$, $y=b$, $z=c$, &c.

5. $$\frac{\lfloor n-3}{\lfloor n-r}r(r-1)(r-2).$$ 6. (1) $\{x^2+y^2+(3-n)xy\}n$.

(2) $2\{n-1+(\frac{1}{2})^n\}$. (3) $1-\frac{1}{3^n}-\frac{n}{3^n}$.

7. $4px^{-3}+4p(2p^2-3k^2)x^{-4}$.

1881. 2. 17 yrs. and 11 yrs. 3. 520. 4. r^3+3r.

5. £7812 10*s*.

1882. 1. (i) $x=2$. (ii) $x=2$ or $\frac{1}{4}\{3\pm\sqrt{(-7)}\}$. 2. $3x-5$.

3. $\frac{22}{3}\sqrt{2}$. 5. 7 times nearly. 6. 2^n.

1883. 1. ·23, $15-5\sqrt{6}+10\sqrt{3}-10\sqrt{2}$.

2. (1) $x=\pm\sqrt{(a^2+b^2+c^2)}$. (2) $x=4\frac{1}{3}$, $y=1\frac{2}{3}$.

6. $$\frac{15\cdot 14\cdot 13\cdot 12\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 5}\times\frac{5\cdot 4}{1\cdot 2};\ \frac{14\cdot 13\cdot 12\cdot 11\cdot 10}{1\cdot 2\cdot 3\cdot 4\cdot 5}\times\frac{5\cdot 4}{1\cdot 2}.$$ 7. 312.

1884. 1. $1\frac{1}{3}$, $1\cdot58$. 2. (1) $x=\frac{1}{2}\left\{a\pm\sqrt{\frac{a^2b-4a}{b}}\right\}$,

$y=\frac{1}{2}\left\{b\pm\sqrt{\frac{b^2a-4b}{a}}\right\}$. (2) $x=2$ or 3, $y=4$, $z=8$ or 2.

4. (1) $\frac{1}{3}(29-2n)$; $\frac{1}{3}(28n-n^2)$. (2) $\frac{2n-1}{2^{n-1}}$; $6-\frac{3}{2^{n-1}}-\frac{2n}{2^{n-1}}$.

1885. 1. $2^{\frac{2}{3}}x^2-2x-2^{\frac{1}{3}}x+2^{\frac{4}{3}}-2^{\frac{2}{3}}+1$. 2. (i) $x=\pm1$ or $\pm\frac{1}{3}$.

(ii) $x=\pm\frac{b+c-a}{2\sqrt{(a^2+b^2+c^2)}}$, $y=\pm\frac{a+c-b}{2\sqrt{(a^2+b^2+c^2)}}$,

$$z=\pm\frac{a+b-c}{2\sqrt{(a^2+b^2+c^2)}}.$$

6. $$\frac{(n+1)(n+2)(n+3)\ldots\ldots\ldots(nr-2n+1)x^{r-1}}{n^{r-1}\times1\cdot2\cdot3\cdot4\ldots\ldots(r-1)a^{\frac{nr-n+1}{n}}}.$$

1886. 1. $\frac{2}{5}\{2\sqrt{(-3)}+3-2\sqrt{6}-3\sqrt{(-2)}\}$; 12.

2. (1) $x=0$, 3 or $\frac{3}{2}\pm\frac{\sqrt{(-23)}}{2}$. (2) $x=\pm\frac{(a-b)^2}{a+b}$, $y=\pm\frac{4ab}{a+b}$.

(3) $x=-1,\ 2,\ \frac{\sqrt{5}-1}{2}\pm\frac{\sqrt{(-10-2\sqrt{5})}}{2}$,

$$-\frac{\sqrt{5}+1}{2}\pm\frac{\sqrt{\{2\sqrt{5}-10\}}}{2},\ \&c.$$

3. $x^2-2\frac{p}{q}x+\frac{4}{q}$. 4. (1) $\frac{n^2-n+1}{n}$. (2) $\frac{\sqrt{3}}{3^{n-1}}$; $\frac{3\sqrt{3}}{2}$. (3) $\frac{60}{16-n}$.

B. C. E. PAPERS. p. 193.

1862. 1. (a) $x=\frac{1}{4}$ or $\frac{1}{4}(1\pm\sqrt{2})$, $y=-\frac{1}{4}(1\pm\sqrt{2})$ or $\frac{1}{4}$.

(b) $x=\pm\frac{3}{2}$ or $\pm\frac{2}{3}$, $y=\pm1$, $z=\pm\frac{2}{3}$ or $\pm\frac{3}{2}$.

(c) $x=10$, $y=\pm6$. 3. $x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}-\frac{1}{2}x^{-\frac{5}{2}}+\frac{1}{4}x^{-\frac{7}{2}}$

$2\sqrt{(-y^2)}-3(x^2+y^2)^{\frac{1}{3}}\left[\{x+\sqrt{(-y^2)}\}^{\frac{1}{3}}-\{x-\sqrt{(-y^2)}\}^{\frac{1}{3}}\right]$.

4. (a) $\{\frac{1}{2}n(n+1)\}^2$. (b) $\frac{2x}{(1-2x)^2}$. 5. Rs 74·297.

1863. 2. $\frac{x^3-2x^2+12x-18}{x^3-2x^2+x+4}$. 3. (1) $x=\frac{cd(a+b)-ab(c+d)}{ab-cd}$.

(2) $x=\pm1$ or $\frac{9}{10}\{-1\pm\sqrt{(-19)}\}$. 5. $\frac{1}{2}r(r+1)a^{-1-2r}x^{-r+1}$

$a^{-3}+3a^{-5}x^{-1}+6a^{-7}x^{-2}+10a^{-9}x^{-3}+15a^{-11}x^{-4}$.

6. 3 miles an hour ; 15 miles.

1864. 1. $2\dfrac{a+b}{a-b}$, $\dfrac{4ab}{a^2-b^2}$, $\dfrac{(a^2+b^2)(a^2+4ab+b^2)}{(a^2-b^2)^2}$

2. (a) $x=18, y=10$. (b) $x=7$ or $-\frac{1}{3}$.

3. (a) $\frac{3}{5}\{1-(-\frac{2}{3})^n\}$. (b) $\dfrac{1}{(1+x)^2}$.

4. 9·7979, 12, 14·6969, 18, 21·9927.

1865. 1. x^2+3x-1. 2. $\frac{3}{5}\{1-(-\frac{2}{3})^n\}$; 16. 3. 2^r.

4. (1) $x=-1\pm\sqrt{5}$ or $-1\pm\sqrt{(37)}$.

(2) $x=\pm 2$ or ± 3, $y=\pm 3$ or ± 2.

5. At 25 minutes after 4.

1866. 1. $(x-2)(x-4)(x-6)$. 3. 3 or 8.

4. $5\frac{11}{32}$; $\dfrac{a^8}{2(4-a^2)}\left\{\left(\dfrac{4}{a^2}\right)^n-1\right\}$.

5. $\dfrac{4}{ax^2}+\dfrac{32}{a^{\frac{5}{2}}x^4}+\dfrac{192}{a^4x^6}+\dfrac{1024}{a^{\frac{11}{2}}x^8}$; $\dfrac{2^{2r}r}{a^{\frac{3r-1}{2}}x^{2r}}$.

6. (1) $x=\frac{89}{26}a$. (2) $x=0, 1, -3$ or -4.

(3) $x=\pm\frac{1}{2}, \pm\frac{1}{3}\sqrt{(-14)}$; $y=\pm\frac{1}{3}, \mp\sqrt{(-\frac{7}{2})}$. 7. $9\frac{5}{8}$ miles.

1867. 1. $\dfrac{a-x}{a+4x}$; $\dfrac{x^2-2x+3}{2x^2+5x-3}$.

3. $9\frac{1}{3}$; -85 ; $\dfrac{1-x^n}{(1-x)^2}-\dfrac{nx^n}{1-x}$. 4. 3 or $-\dfrac{11}{2}$.

5. (1) $x=0$ or $-\frac{3}{2}$. (2) $x=2$ or $5\frac{1}{3}$.

(3) $x=3$ or $-\frac{1}{3}$. $y=-6$ or $\frac{2}{3}$.

6. $n=-8$. $1+25x+250x^2+1250x^3+3125x^4+3125x^5$;

$1+2x+3x^2+4x^3+\ldots\ldots$

1868. 1. (1) $\dfrac{2a+36a^{\frac{1}{3}}b^{\frac{2}{3}}}{a-27b}$. (2) $\dfrac{(a-b)^2}{a^2+b^2}$.

2. x^2-4 ; $(x-3)(x^3-x^2-4x+4)$.

4. (1) $x=3$ or 4, $y=4$ or 3.

(2) $x=5$ or 13 ; $y=7, 6, \frac{1}{2}\{5\pm\sqrt{(-143)}\}$.

$z=6, 7, \frac{1}{2}\{5\mp\sqrt{(-143)}\}$.

5. $\frac{1}{2}(9n-n^2)$; $\dfrac{a^2}{a^2-2}\left\{1-\left(\dfrac{2}{a^2}\right)^n\right\}$; 1 or 8. 7. 60.

8. $120a^7$; $1-\frac{1}{4}x-\frac{1}{32}x^2-\frac{1}{128}x^3-\frac{5}{2048}x^4$;
$$-\frac{1\times1\times3\times5\ldots(5-2r)}{2^{r-1}\times1\cdot2\cdot3\cdot4\ldots(r-1)}\left(\frac{x}{2}\right)^{r-1}.$$

1869. 1. 35. 2. (1) $\dfrac{x^2+x-2}{x^2-x+2}$, $(x^2-y^2)^3$.

4. (1) $x=\dfrac{a+b\pm\sqrt{(b^2-3a^2-2ab-4ac)}}{2a}$; (2) 5 or $\frac{13}{10}$.

(3) $x=\dfrac{c(a^2+b^2)}{a^2-b^2}$, $y=\dfrac{c(a^2+b^2)}{2ab}$.

(4) $x=74$, $y=-349$, $z=-420$. 5. 8 ; 3, 2, 1 and 0.

6. (1) $\dfrac{a^{n+1}-y^{n+1}}{a-y}$. (2) $\frac{3}{4}\{1-(\frac{1}{3})^n\}-\dfrac{n}{2\times3^n}$.

7. $19\times17\times13\times44a^{10}b^{10}$;
$a^{-2}+a^{-4}x^2+a^{-6}x^4+a^{-8}x^6+a^{-10}x^8$. $a^{-2r}x^{2r-2}$.

1870. 1. $\dfrac{3x^2+4}{2x^2-3x+4}$. $(x-2)(2-1)(x^3-8)$.

2. x^2. 3. (a) $x=0$ or $-\frac{1}{9}$. (b) $x=9$ or 1, $y=1$ or 9.
(c) $x=\pm4$, $y=\pm2$, $z=\pm2$ &c. 5. (a) -280.
(b) $\frac{31}{32}\{\sqrt{2}+1\}$. (c) $\frac{9}{4}\{1-(-\frac{1}{3})^n\}$.

6. $10a^{-13}x^9$; $-6^7\times7x^5y^5$.

1871. 1. $x-1$. 3. (i) $x=\frac{40}{37}$. (ii) $x=2$, $y=2$.
(iii) $x=10$, $y=5$ or -13, $z=3$ or -15.

5. 9 or 18. $\dfrac{4}{3\sqrt{2}}\{1-(\frac{1}{3})^{10}\}$. $\dfrac{(n+1)^2}{2n^2+n}$.

6. $\dfrac{x^6}{a^6}-\dfrac{6x^4}{a^4}+\dfrac{15x^2}{a^2}-20+\dfrac{15a^2}{x^2}-\dfrac{6a^4}{x^4}+\dfrac{a^6}{x^6}$;
$a^{-3}+3a^{-4}x+6a^{-5}x^2+10a^{-6}x^3+15a^{-7}x^4+21a^{-8}x^5+$ &c.
$13\times110\times9\times y^4x^{-4}$.

1872. 1. (1) $abxy$. (2) $\dfrac{x^3+x}{x^4+1}$. x^2+5x+6.

2. (1) $x=\frac{3}{2}$. (2) $x=9$, $y=\frac{1}{9}$; (3) $x=1$, $y=\sqrt{2}$, $z=\sqrt{3}$.

3. 20. 4. 2, etc. ; 0 ; $\dfrac{255}{64}\sqrt{2}$. 5. 90.

6. $a-\frac{1}{3}a^{-2}x^3-\frac{1}{9}a^{-5}x^6-\frac{5}{81}a^{-8}x^9-\frac{10}{243}a^{-11}x^{12}$.

1873. 1. (1) $\frac{1}{(a-b)^2}$. (2) $\frac{3x^2-yx}{2x^2-y^2}$. (3) $\frac{1}{a+b}$.

2. (1) $x=\pm 6$. (2) $x=\pm 3$ or $\pm 5\sqrt{\frac{1}{2}}$; $y=\pm 2$ or $\pm\sqrt{\frac{1}{2}}$.

(3) $z=\left(\frac{c^3}{ab}\right)^{\frac{1}{4}}$, $y=\left(\frac{b^3}{ac}\right)^{\frac{1}{4}}$, $x=\left(\frac{a^3}{bc}\right)^{\frac{1}{4}}$.

3. $9\frac{17}{72}$. $10\frac{1}{3}$. 4. 1585; 562.

5. $32x^{\frac{10}{3}}-240x^{\frac{8}{3}}y^{\frac{3}{2}}+720x^2y^3-1080x^{\frac{4}{3}}y^{\frac{9}{2}}+810x^{\frac{2}{3}}y^6-243y^{\frac{15}{2}}$; -1; 9·99666444.

1874. 1. $\frac{1+p}{2pq}$. 2. (1) -1; 1. 3. (1) $x=3$ or 5.

(2) $x=\frac{x}{a+b}$, $y=\frac{b}{a+b}$.

4. $x=1, 2, 3$; $y=2, 3, 1$; $z=3, 1, 2$.

5. $n=5$. 6. $\frac{\lfloor 2n}{\lfloor n \lfloor n}$.

1875. 1. x^2-2x+2. $x^{\frac{m}{2}}+x^{-\frac{m}{2}}$.

2. (1) $x=\frac{a(m+n)-2ap}{p(m+n)-2mn}$.

(2) $x=\{\frac{1}{2}[a-2\pm\sqrt{(a^2-2a)}]\}^n$.

(3) $x=0$ or $2a$; $y=\pm b$; $z=\pm c$.

5. 15 or 16. 6. 52. 5·0990.

1876. 1. 2; $\frac{2x^3+x^2}{x^4-1}$. 2. $x-2$; $(2x+1)(3x-2)(7x-1)$.

3. (1) $x=6$, -7 or $-\frac{1}{2}\{1\pm\frac{1}{2}\sqrt{(221)}\}$.

(2) $x=16$ or -8; $y=8$ or -16.

(3) $x=b\pm\sqrt{(a^2-b^2+c^2)}$, &c.

4. $\frac{\lfloor 14}{\lfloor 8 \lfloor 6}$, $\frac{\lfloor 14}{\lfloor 9 \lfloor 5}$. 6. $1+4x+2x^2-8x^3-22x^4$.

1877. 1. $y(y^2+2)(y^4+4y^2+2)\sqrt{(y^2+4)}$; 1; 6.

2. (1) $x=\frac{1\pm\sqrt{(1+a^2)}}{a}$. (2) $x=4$ or -2, $y=2$ or -4, etc.

4. $\frac{\sqrt{(10)^n}-1}{\sqrt{(10)}-2}$. $2+10+24......$to n terms$=n^2(n+1)$.

5. $\left(\dfrac{\lfloor 10}{\lfloor 5 \lfloor 5}\right)^2$ 6. $a^{-4}-2a^{-6}x^2+3a^{-8}x^4$;
$-4a^{-10}x^6+5a^{-12}x^8-6a^{-14}x^{10}+\ldots\ldots$; 3^7.

1878. 1. 1.

2. (1) $x=\dfrac{a^2-b^2\pm\sqrt{\{(a^2-b^2)^2+4(a^2+3ab)(b^2+3ab)\}}}{2(a^2+3ab)}$

(ii) $x=5$ or $\frac{1}{6}$; $y=6$ or $\frac{1}{5}$, etc.

(iii) $x=\pm\sqrt{\left\{\frac{1}{2}\times\dfrac{a^2-(b-c)^2}{b+c-a}\right\}}$; &c.

3. 9; 19. 4. $\lfloor 6 \lfloor 3$.

5. Value of diamonds £120 and that of rubies £270.

6. $\dfrac{(1-n)(1-2n)\ldots\ldots(1-nx+n)}{\lfloor x\, n^x}x^x(-1)^x$.

1879. 1. $(x^4+y^4-x^2y^2\sqrt{2})(x^4+y^4+x^2y^2\sqrt{2})$;
$(x-y)(x+y)(x^2+y^2)(x^2-xy+y^2)(x^2+xy+y^2)(x^4-x^2y^2+y^4)$;
$(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)(x^8-x^4y^4+y^8)$.

2. (1) $x=\pm a\sqrt{(-\frac{1}{3})}$, etc.

(2) $(a^2+b^2+c^2-ac-bc-ab)x=8abc-(a+b)(b+c)(c+a)$.

(3) Multiplying the equations and by substituting, we get, $x^2y^2z^2(a^3+b^3+c^3+2xyz)=a^3b^3c^3$, a cubic equation in xyz.

3. (1) $\frac{25}{16}$. (2) $\frac{1}{6}n(n+1)(2n+1)$. 4. 2^7-1.

5. $a^{\frac{4}{7}}-\frac{4}{7}a^{-\frac{3}{7}}x-\dfrac{6}{7^2}a^{-\frac{10}{7}}x^2-\dfrac{20}{7^3}a^{-\frac{17}{7}}x^3-\dfrac{85}{7^4}a^{-\frac{24}{7}}x^4$.

$\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}=\dfrac{\sqrt{(a+x)}}{\sqrt{(a-x)}}\times\dfrac{\sqrt{(a+x)}}{\sqrt{(a+x)}}=\dfrac{a+x}{\sqrt{(a^2-x^2)}}$

$=(a+x)(a^2-x^2)^{-\frac{1}{2}}$

$=1+a^{-1}x+\frac{1}{2}a^{-2}x^2+\frac{1}{2}a^{-3}x^3+\frac{3}{8}a^{-4}x^4+\frac{3}{8}a^{-5}x^5+\ldots\ldots$

The 8*th* term is the first negative term; $-\dfrac{17\times13\times x^7}{2^{11}+3^6}$; 5·03159.

1880. 2. $(7x+32)(x-5)$, $(nx-m)(mx+n)$.

3. (i) $x=\dfrac{b(c+a-b)}{a}$. (ii) $x=\pm2, 3$ or $-\frac{7}{4}$; $y=0$ or $-\frac{5}{4}$.

(iii) $x=y=z=a^2+b^2+c^2-ab-ac-bc$.

5. $\dfrac{(4n+1)4n(4n-1)\ldots\ldots(3n+1)}{1\cdot2\cdot3\ldots\ldots\ldots(n+1)}$ 6. $5\frac{1}{3}$ or $5\cdot\dot{3}$.

1881. 1. $\dfrac{x+3}{x-2}$. 2. (1) $x=5$ or $5\pm\sqrt{(10)}$.

(2) $x=3$, $y=5$ or $-\frac{7}{2}$, $z=7$ or -10. 3. $x^4+x^2+1=0$.

4. The expression may have any positive value, but of the negative values it must not have a greater value than $\frac{2}{3}$.

6. $\dfrac{\lfloor 11}{\{\lfloor 3\lfloor 2\}^2}$; $\dfrac{\lfloor 9}{\lfloor 3\lfloor 2\lfloor 2}$; $\dfrac{\lfloor 8}{\lfloor 3\lfloor 2\lfloor 2}$.

7. $3^7-14x\times3^6+84x^2\times3^5-280x^3\times3^4+560x^4\times3^3$—

$$3^{-7}\left\{1+\frac{14}{3}x+\frac{112}{9}x^2+\frac{224}{9}x^3+\frac{1120}{27}x^4+\ldots\ldots\right\}.$$

9. (1) $\dfrac{2^n-(-1)^n}{3\times2^{n-1}}$. (2) $\frac{1}{4}(n+3)(n+2)(n+1)n$.

1882. 1. (a) $x=-a$, $y=-b$.

2. $x=\dfrac{(nm_1-mn_1)d}{\sqrt[3]{\{(nm_1-mn_1)^3+(ln_1-l_1n)^3+(ml_1-m_1l)^3}}$.

3. $\dfrac{1}{(1-x)^2}$. 4. Rs18 per 1000 bricks. 5. 78.

6. $\frac{1}{3}n(n+1)(n+2)$. 7. $ad-bc$. 8. 0.

1883. (F. C. E.) 1. $\dfrac{ab+bc+ac}{bc+ac-ab}$. 2. $x=a$ or b.

3. $\dfrac{\lfloor 9}{\lfloor 2\lfloor 3\lfloor 4}$. 5. See page 139, Ex. 12; $\dfrac{2^n(2n-1)(2n-3)\ldots3\cdot1}{1\cdot2\cdot3\cdot4\ldots n}$.

1883. (B. C. E.) 1. $x^2-\frac{1}{2}x-\dfrac{2}{x}$; $\sqrt{\dfrac{1+\sqrt{(1-x^2)^{-1}}}{2}}$

$$+\sqrt{\frac{1-\sqrt{(1-x^2)^{-1}}}{2}}.$$

2. (1) $x=5$. (2) $x=1$ or $\frac{1}{2}$. (3) $x=\dfrac{1}{n}$, $y=\dfrac{1}{m}$.

3. 8 : 9. 4. $-\dfrac{57}{128}$; $\dfrac{(a^2-x^2)(a+x)}{a+x-1}$.

5. $-\dfrac{(2x+1)(2x)(2x-1)(2x-2)(2x-3)}{1\cdot2\cdot3\cdot4\cdot5}x^{2x-9}$.

6. $\bar{2}\cdot2108533$. 7. 36 yds.

1884. (F. C. E.) (1) $x=2\sqrt{\dfrac{a^2-4a+3}{3}}$

(2) $x=\pm 4$ or $\pm\dfrac{10}{\sqrt{3}}$, $y=\pm 3$ or $\pm\dfrac{9}{\sqrt{3}}$, $z=\pm 2$ or $\pm\dfrac{8}{\sqrt{3}}$.

2. $\dfrac{(a+b)^n-(a-b)^n}{2b(a+b)(a-b)^{n-2}}$; 3 or -11. 4. 85.

5. $2^{-6}\left\{1+\dfrac{3}{2}x+\dfrac{3^3}{20}x^2+\dfrac{7\cdot 3^2}{200}x^3+\dfrac{7\times 3^4}{1000}x^4+\dfrac{14\cdot 9\cdot 3^5}{100000}x^5\right\}$;

x^r ; $\frac{1}{2}(3n^2+3n+2)$.

1884. (B. C. E.) 1. $\dfrac{(a^2+bc\sqrt{3})\sqrt{(ab)}}{(a-c)\sqrt{3}}$; $\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}+\dfrac{1}{2}$.

2. ae^x+e^x+a+1. 3. $x=\dfrac{a(n-1)^2}{2n-1}$;

for $\dfrac{2ax(x^2+a^2)^{\frac{1}{2}}}{a-(x^2+a^2)^{\frac{1}{2}}}$ read $\dfrac{ax(x^2+a^2)^{\frac{1}{2}}}{a+(x^2+a^2)^{\frac{1}{2}}}$, $x=0$ or $\dfrac{2a^3}{1-a^4}$

4. (1) $x=\pm\sqrt{\left\{a^2-\left(-h^2\pm\sqrt{\dfrac{2a+h^4}{2}}\right)^4\right\}}$.

(2) $x=1$ or 625, $y=625$ or 1.

5. $x^3-(5+\sqrt{2})x^2+(11+4\sqrt{2})x-7(\sqrt{2}+1)=0$.

7. $1-x+\dfrac{m+1}{1\cdot 2}x^2-\dfrac{(m+1)(2m+1)}{1\cdot 2\cdot 3}x^3$

$+\dfrac{(m+1)(2m+1)(3m+1)}{1\cdot 2\cdot 3\cdot 4}x^4-\ldots\ldots$

1885. 1. 1; x^2-x-1. 2. (i) $x=\pm a$ or $\pm\dfrac{1}{a}$

(ii) $x=16$ or 4, $y=9$ or -3.

3. nth term $=3n-2$, last term $=6n+1$.

4. $3\times 2^{\frac{15}{2}}$, $3^2\times 2^6$, $3^3\times 2^{\frac{9}{2}}$, $3^4\times 2^3$, $3^5\times 2^{\frac{3}{2}}$.

5. 13 : 3. 7. $\dfrac{4\times 7\times 9\times\ldots\ldots(3r-5)}{3^{r-1}\ \underline{|r-1}}x^{2r-2}$.

8. $\bar{1}\cdot 5997802$; $\cdot 8856217$.

Answers to H. Ex. Questions. p. 231.

7. $1166\frac{2}{3}$, 1169, 1000, 1002.

10. $(a^2b^2+b^2c^2+a^2c^2)(a^2bc+\beta^2ac+\gamma^2ab)$
$$=a^2b^2c^2(ab+ac+bc).$$

12. $-3, 4, -6$. 13. 28. 14. $\dfrac{1}{(1-x)^2}$; $\frac{1}{6}(r+1)(r+2)(r+3)$.

15. (1) $x=3$, -1 or $\frac{1}{2}\{-1\pm\sqrt{(76)}\}$.

(2) $x=\frac{1}{2}$ or $\frac{1}{3}$, $y=\frac{1}{3}$ or $\frac{1}{2}$. 18. $\dfrac{\lfloor 52}{(\lfloor 13)^4}$.

19. $\frac{3}{4}\{(2n-1)3^n+1\}$. 20. $\frac{3}{4}$. 21. $x=10$.

22. (a) $a^2a_1^2x^2-4aba_1b_1x+2(acb_1^2+a_1c_1b^2)-aa_1cc_1=0$.

23. $\frac{1}{2}P^2$. 24. $c=\sqrt{(a+1)}$.

25. 4369. 39. 11·005506.

Appendix. p. 317.

1. 48. 2. 120. 3. 540 *nishcas.* 4. 15.
5. 29, 21. 6. 16. 7. 100. 8. 72.
9. 576. 10. 24, 28 and 42. 11. 12 cubits.
12. 12 cubits. 13. $3\frac{3}{4}$ cubits 14. 15 and 8.
15. 12 and 5. 16. 6 cubits.
17. 96, 64, 4 and 480 *nishcas.* 18. 12. 19. 14, 7 and 29.
20. 5, $\frac{1}{2}\{1\pm\sqrt{(-27)}\}$. 21. 16 or 48. 22. 50.
23. 12, 35 and 37. 24. 8, 15 and 17. 25. 24, 7 and 25.
26. 59. 27. 42, 33 and 28. 28. 143. 29. 10, 20, 30 &c.
30. 22 and 27; 48; 196. 31. 18. 32. 2186.
33. The doors may be opened 255 times. 34. 63.
35. 60480. 36. $\lfloor 10$, $\lfloor 4$.

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Head Master, Zilla School, Jessore.

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